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The Application and Extension of Group Theory in Rubik's Cube

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Abstract. The paper aims to explore the practical application of group theory in the analysis of Rubik's Cube. As a highly challenging and intellectual three-dimensional puzzle, the Rubik's Cube has attracted widespread research interest. By applying the concepts and techniques of permutation groups in group theory, we further explore the properties of Rubik's Cube transformations. Firstly, we introduce the basic concepts of group theory and the definition of permutation groups, as well as the background and objectives of the Rubik's Cube problem. Then, we discuss in detail how to convert Rubik's Cube operations and rotations into elements and operations of permutation groups, to analyze and solve Rubik's Cube problems within an abstract algebraic framework. We explain the properties and structure of permutation groups, such as group order, subgroups, and generators, as well as how to use these concepts to simplify the process of solving Rubik's Cube. We also introduce some classical group theory techniques and algorithms, such as inverse elements, exponentiation, and cycle notation, and explain their practical application in Rubik's Cube analysis.

Keywords: Group Theory; Cude; Permutation Groups.

1. Introduction

The Rubik's Cube(Figure.1) has been a subject of study for mathematicians who have discovered that it encompasses various mathematical concepts. It gained global attention from mathematicians as early as the 18th International Congress of Mathematicians held in Helsinki, Finland, in the late 19th century[1–4].

Fig 1. Rubik's Cube

As a symbol of intelligence and fashion, playing Rubik's Cube has evolved from a simple puzzle game to a diverse competitive sport, requiring coordination of hands, eyes, and brain. With the efforts of Rubik's Cube associations around the world, various competitive events have been developed, such as speedcubing, one-handed solving, blindfolded solving, and fewest moves solving. The rules for Rubik's Cube competitions have been continuously improved, and world records continue to amaze people. The development of different types of Rubik's Cubes has also been remarkable. In addition to the most common 3x3x3 Rubik's Cube, there are various sizes and shapes, including 2x2x2, 4x4x4, 5x5x5, and more. Rubik's Cube enthusiasts are constantly exploring and innovating from different aspects, such as mechanisms, sizes, and shapes. Group theory concepts play a crucial role in understanding the structure and properties of Rubik's Cubes. The transformations and movements of Rubik's Cubes can be represented as elements and operations in permutation groups, allowing for analysis and solution of Rubik's Cube problems in an abstract algebraic framework. By applying group theory principles, solving algorithms can be optimized and new solving techniques can be discovered.

Many concepts from group theory in mathematics, such as actions, permutations, transitivity, conjugation, and permu- tations, find application in understanding the mathematical transformations and patterns in the Rubik's Cube. Scholars from various disciplines have analyzed the operational principles of the Rubik's Cube. For instance, reference [5] provides a brief overview of the structures of Rubik's Cubes in the forms of tetrahedrons, hexahedrons, octahedrons, dodecahe- drons, and icosahedrons, along with the associated knowledge. Reference[6] presents the structure of different components based on the approach for restoration, but does not provide an overall structure and lacks a supporting operation. The paper [7] characterizes the group structure of the rotational movement of a Rubik's Egg, which has a topological structure equivalent to that of a regular octahedron.

This paper aims to delve deeper into the mathematical transformations and models of the Rubik's Cube, utilizing relevant mathematical knowledge to investigate its mathematical transformation patterns and extend the Rubik's Cube group to higher-order cubes.

2. Group Theory in Rubik's Cube

The Rubik's Cube is a cube made up of 26 small cubes, with 6 faces, each containing 9 small squares. The small cubes in the Rubik's Cube can be classified into three categories based on their positions: corner cubes, which are located at the 8 corners; edge cubes, which are located in the middle of each edge; and center cubes, which are located in the center of each face. It is obvious that the Rubik's Cube has 6 center cubes, 8 corner cubes, and 12 edge cubes.

A state of the Rubik's Cube refers to a configuration that maintains the shape of a cube. In the initial state, the colors of the small squares on each face of the Rubik's Cube are the same. A single rotation that maintains the shape of the cube is called a move. When the Rubik's Cube is placed in front of an observer, there are 6 basic moves in all possible moves:

1 U, which rotates the top face clockwise (when viewed from top to bottom); D, which rotates the bottom face clockwise (when viewed from bottom to top);

2 L, which rotates the left face clockwise (when viewed from left to right); R, which rotates the right face clockwise (when viewed from right to left);

3 F, which rotates the front face clockwise (when viewed from front to back); B, which rotates the back face clockwise (when viewed from back to front).

These 6 moves, U, D, L, R, F, B, are considered the basic moves of the Rubik's Cube.

Let F_1 and F_2 be moves, and let a, b, c be three states of the Rubik's Cube. If F_1 transforms state a into state b, and F_2 transforms state b into state c, defined asF_2 (F_1 (a)) = c Clearly, any move of the Rubik's Cube can be composed of the 6 basic moves U, D, L, R, F, B.

Definition 1 A group G is a set equipped with an operation that combines any two elements of the set to produce a third element, satisfying four axioms: closure, associativity, identity element, and inverse element for each element.

Definition 2 A subgroup is a subset of a group that is itself a group under the same operation.

Definition 3 A cyclic group is a group where every element can be generated by repeatedly applying the group operation to a single element, called the generator.

Definition 4 Given a subgroup H of a group G, a left coset of H is a set of the form gH, where g is an element of G. A right coset of H is a set of the form Hg.

Definition 5 An isomorphism is a bijective mapping between two groups that preserves the group structure, namely, the operation. If there exists an isomorphism between two groups, we say they are isomorphic.

Definition 6 A subgroup H of a group G is considered normal if, for every element g in G, the left coset gH is the same as the right coset Hg.

Definition 7 Given a normal subgroup H of a group G, the set of all cosets of H in G forms a group under the operation defined as (gH)(kH) = (gk)H. This group is called the quotient group of G by H.

Definition 8 The group generated by the synthesis operation of six basic moves G =< U, D, L, R, F, B > is called the Rubik's Cube group.

Definition 9 A commutative or abelian group G is a set equipped with an operation that combines any two elements of the set to produce a third element, satisfying the four axioms: closure, associativity, identity element, and inverse element for each element. Additionally, in a commutative group, the operation is commutative, meaning that for any elements a and b in G, the result of combining them is the same regardless of the order, such as a * b = b * a.

Definition 10 A group G is said to be transitive on a set X if, for any pair of elements x_1 x_2 in x there exists an element g in G such that $x_2 = g * x_1$. In other words, the action of G on X is transitive if every element in X can be mapped to any other element under the group operation.

Due to the fact that the center blocks of each face do not change position when any of the six faces are rotated. Therefore, the initial state"a" corresponds to the natural arrangement of the digits 1, 2, ..., 48, as shown in Figure 2.

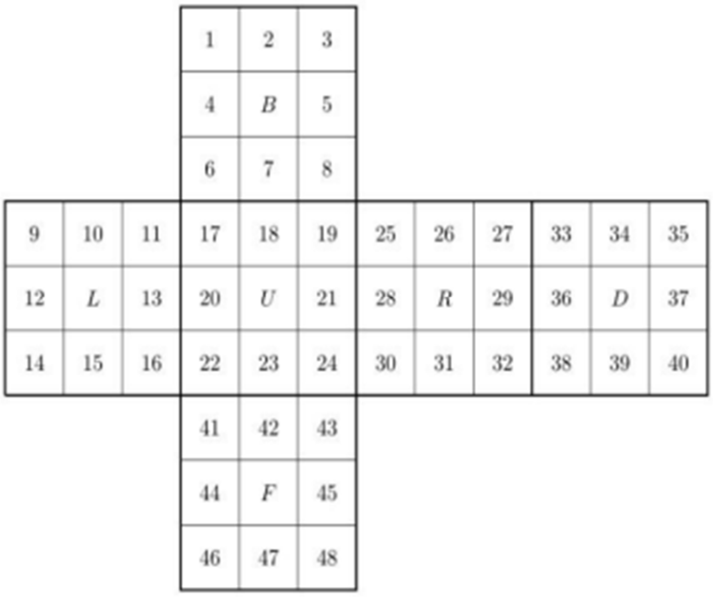

Fig 2. 3rd order Rubik's Cube expansion diagram

Definition 11 A permutation is a one-to-one correspondence on a set A. The set of all permutations on A is denoted as S_n and is called the symmetric group of order n. Each subgroup of S_n is called a permutation group.

Any permutation $\rho \in S_n$ can be written in the following form:

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} \tag{1}$$

where i_1 , i_2 , $\cdots$ i_n is an arrangement.

Theorem 1 The Rubik’s Cube Group G is isomorphic to a subgroup of S_{48} .

Use u, d, f, b, r, l to represent each face and its corresponding center block. For small faces on corner blocks, use xyz to represent their orientation: meaning the small face with the orientation of x and yz.

Theorem 2 The disjoint cycles in S_n are commutative.

Corollary 1 In the Rubik’s Cube group, the rotations of opposite sides are commutative.

Proof 1 In the Rubik’s Cube group G =< U, D, L, R, F, B > , the rotations of opposite sides are disjoint. Therefore, $UD = DU$, $FB = BF$, $LR = RL$.

$$\begin{gathered}
U = (ulb\ ubr\ urf\ \ ufl)(ub\ ur\ uf\ \ ul)(bul\ rub\ fur\ luf)(bu\ ru\ fu\ lu)(bru\ rfu\ flu\ lbu) \\
D = (dbldlf\ \ dfrdrb)(db\ dldf\ \ dr)(bld\ lfdfrd\ rbd)(bd\ ldfd\ rd)(bdr\ ldb\ fdl\ rdf) \\
F = (flu\ fur\ frd\ fdl)(fu\ frfd\ fl)(ufl\ \ rfudfr\ lfd)(uf\ \ rf\ df\ lf)(urf\ rdf\ dlf\ luf) \\
B = (bulbld\ bdr\ bru)(bubl\ bdbr)(ulb\ ldb\ drb\ rub)(ub\ lb\ db\ rb)(ubr\ lbudbl\ rbd) \\
L = (luf\ \ lfd\ ldb\ lbu)(lu\ lf\ \ ld\ lb)(ufl\ fdl\ dbl\ bul)(ulfldl\ bl)(ulb\ flu\ dlf\ \ bld) \\
R = (rfu\ rub\ rbd\ rdf)(ru\ rb\ rd\ rf)(urf\ \ bru\ drbfrd)(urbr\ dr\ fr)(ubrbdr\ dfr\ fur)
\end{gathered} \tag{2}$$

F* represents the set of small faces of the Rubik’s Cube, B* represents the set of blocks of the Rubik’s Cube, E_F represents the set of small faces of the Rubik’s Cube edge blocks, V_F represents the set of small faces on the Rubik’s Cube corner blocks, E_B represents the set of edge blocks of the Rubik’s Cube, and V_B represents the set of corner blocks of the Rubik’s Cube, we have:

$$F_* = E_F{}^{\mathbf{n}}\ V_F, E_F \cup V_F = \mathrm{Q} B_* = E_B{}^{\mathbf{n}}\ V_B, E_B \cup V_B = \mathrm{Q}$$

Theorem 3 The action of the Rubik’s Cube group in V_B is transitive.

Proof 2 Taking the *dfl* corner block as an example, consider the following rotation M = FFFDBBBD^{-1} , then the corner block traverses all the corner blocks along the above path, and finally returns to the starting point. Any two elements in the cold can be transferred under the action of G, so the role of G in V_B is transferred.

Definition 12 When n is an even number, N = $\frac{n}{2}$; when N is an odd number, N = $\frac{n+1}{2}$. f_i is the i-th face from front to back. Fi represents the clockwise rotation of the f_i by 90 degrees. b_i is the i-th face from back to front. Bi represents the clockwise rotation of the bi by 90 degrees. Generalized Rubik’s Cube GroupL_n is generated by F_i , B_i , R_i , L_i, U_i , $D_i(i\ \mathrm{e}\ Z,\ 1\ \equiv\ i\ \equiv\ N)$.

The Rubik’s Cube has 43,252,003,274,489,856,000 possible states. It was a long-standing puzzle to determine the minimum number of moves required to solve any configuration of the Rubik’s Cube, also known as “God’s Number”. For many years, researchers and enthusiasts worked on solving this problem until, in recent years, it was proven with the assistance of computer analysis and algorithms[8, 9]. The resulting God’s Number, which represents the minimum number of moves needed to solve any scrambled Rubik’s Cube, stands as a testament to the power of computational analysis in unraveling complex puzzles. Scholars’ research is shown in the table below[10]:

Table 1. Scholars' research

date	infimum	supremum	researcher
1981.07	18	52	Morwen Thistlethwaite
1990.12	18	42	Hans Kloosterman
1992.05	18	39	Michael Reid
1992.05	18	37	Dik Winter
1995.01	18	29	Michael Reid
1995.01	20	29	Michael Reid
2005.12	20	28	Silviu Radu
2006.04	20	27	Silviu Radu
2007.05	20	26	Dan Kunkle, Gene Cooperman
2008.03	20	25	Tomas Rokicki
2008.04	20	23	Tomas Rokicki
2008.08	20	22	Tomas Rokicki
2010.07	20	20	Tomas Rokicki, Herbert Kociemba, Morley Davidson, John Dethridge

Table 2. The results

	Generators	Size	Factorization
1	U	4	2^2
2	U, RR	14400	$2^6 \cdot 3^2 \cdot 5^2$
3	U, R	73483200	$2^6 \cdot 3^8 \cdot 5^2 \cdot 7$
4	RRLL, UUDD, FFBB	8	2^3
5	RI, Ud, Fb	768	$2^8 \cdot 3$
6	RL, UD, FB	6144	$2^{11} \cdot 3$
7	FF, RR	12	$2 \cdot 3^2$
8	FF, RR, LL	96	$2^5 \cdot 3$
9	FF, RR, LL, BB	192	$2^6 \cdot 3$
10	FF, RR, UU	2592	$2^5 \cdot 3^4$
11	FF, RR, LL, UU	165888	$2^{11} \cdot 3^4$
12	FF, BB, RR, LL, UU	663552	$2^{13} \cdot 3^4$
13	FF, BB, RR, LL, UU, DD	663552	$2^{13} \cdot 3^4$
14	LLUU	6	$2 \cdot 3^2$
15	LLUU, RRUU	48	$2^4 \cdot 3$
16	LLUU, FFUU	1296	$2^4 \cdot 3^4$

The Rubik’s Cube has so many rotational states, which is very complex, but what if there are some restrictions these rotational rules? For an extreme example, the state generated by only rotating R is very obvious, with only 4 types. For example, if you can only rotate from 90 ° per layer to 180 ° per layer, there are many states generated through this operation, but compared to the initial numbers, they are also very few. And this messy Rubik’s Cube doesn’t look so messy either, because the color direction is correct. For the Rubik’s Cube group and its subgroups, we can use software (https://www.gap-system.org/Doc/Examples/rubik.html) to calculate the order of this group. The results are shown in the table 2.

The group reduction method is to gradually limit the rotation rules of each stage, thereby reducing the state of each stage. That is to gradually degrade the group where the Rubik’s Cube is located to smaller subgroups, and finally to the reduced state, which is backed by rich knowledge of group theory.

The principle of using group theory to solve the Rubik’s Cube is based on the concept of cosets and the coset decomposition algorithm. In group theory, a group is a set of elements with a defined operation (such as the rotations of a Rubik’s Cube) that satisfy certain properties[11–13]. To solve the Rubik’s Cube using the coset decomposition method, we first define a subgroup called the ”stabilizer subgroup.” This subgroup consists of the operations that do not change the cube’s overall structure, such as rotations of individual layers or specific combinations of moves. By applying the coset decomposition algorithm, we partition the set of all cube configurations into different cosets based on the elements of the stabilizer subgroup. Each coset represents a unique combination of moves that can be applied to the cube. The goal is to find a coset that contains the solved cube configuration. By exploring different cosets and their corresponding moves, we can systematically search for a sequence of moves that will lead to the desired coset. This approach allows us to solve the Rubik’s Cube by breaking down the problem into smaller, more manageable subproblems, and applying group theory principles to find the optimal solution. Through the application of group theory, the coset decomposition method provides a systematic and efficient approach to solving the Rubik’s Cube, utilizing the properties and structure of group elements to determine the minimum number of moves needed for a successful solution.

The coset decomposition method provides several advantages for solving the Rubik’s Cube:

1 Systematic Approach: The method breaks down the problem of solving the Rubik’s Cube into smaller, more manageable subproblems. By dividing the set of all cube configurations into cosets based on a specific subgroup, we can systematically explore different subproblems and their corresponding moves.

2 Optimal Solution: The coset decomposition method allows us to find the optimal solution to the Rubik’s Cube. By leveraging the properties of group elements, the method minimizes the number of moves required to reach a solution. This optimization is achieved through analyzing and manipulating the cosets of the subgroup.

3 Efficiency: Compared to other methods, such as brute-force algorithms, the coset decomposition method provides a more efficient approach. By utilizing group theory principles, the method reduces the number of configurations that need to be explored, effectively reducing the time and effort required to solve the Rubik’s Cube.

The coset decomposition method involves various group theory concepts, including the concept of subgroups, cosets, and the coset decomposition algorithm.

3. Conclusion

The Rubik’s Cube holds a profound significance and has gained immense popularity due to its underlying concepts and the application of group theory. The cube represents a physical manifestation of a mathematical puzzle, providing intellectual stimulation and entertainment. At its core, the Rubik’s Cube embodies the principles of group theory, specifically the concept of a permutation group. The cube's various moves, rotations, and combinations can be represented as elements of a group, where each configuration corresponds to a unique permutation of the cube's colored stickers. Group theory allows us to analyze and understand the cube's behavior, providing a systematic approach to solving it. Group theory provides valuable insights and techniques for solving the Rubik's Cube efficiently. Algorithms derived from group theory help strategize the cube-solving process, enabling enthusiasts to follow a systematic series of moves to solve it. By understanding the cube's inherent group structure and applying group theory principles, individuals can develop problem-solving skills, logical reasoning, and spatial awareness. This mathematical framework also fosters perseverance, patience, and critical thinking, as solving the Rubik's Cube often requires experimentation, evaluation, and adaptation of strategies.

Moreover, the Rubik's Cube's popularity can be attributed to its ability to captivate and engage people across var- ious age groups and cultures. Its challenging nature, combined with the satisfaction of solving it, creates a sense of accomplishment and fosters a sense of community among enthusiasts. Cubing competitions and online communities have flourished, allowing individuals to share their strategies, techniques, and experiences, further enhancing the cube's appeal and promoting the growth of group theory knowledge.

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