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Research on p order nonlinear half wave Schrödinger equations

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Abstract: The objective of our research is to scrutinize the presence and singularity of the comprehensive potent solution for the half-wave Schrödinger equation characterized by p order. Take into account the nonlinear half-wave Schrödinger equations represented by p order:$(i\,\partial_t+\partial_x^2-|D_y|)u=\pm|u|^{p-1}u$. We apply the Brezis-Gallouet style inequality to attain a logarithmic form of regulation. These rationales will be pivotal in validating our primary theorem within Global Well-Posedness. When discussing global Solutions, we infer the solutions for Schrödinger equations within the $\mathcal{H}^s$ realm when $\frac{1}{2}\leq s\leq 1$. Specifically, when $\frac{1}{2}<s\leq 1$, we leverage the Strichartz estimations alongside inequalities following the Brezis-Gallouët pattern to ascertain the well-posedness of Schrödinger equations within the $\mathcal{H}^s$ framework. To elucidate the global well-posedness of Schrödinger equations in the energy subspace $\mathcal{H}^{\frac{1}{2}}$, we undertake a traditional tactic to craft the frail solution within the $\mathcal{H}^{\frac{1}{2}}$ realm, succeeded by the application of a rationale found in existing academic literature to verify the singularity of the frail solution. The coherent progression of the frail solution is derived from the sustained conservation of both mass and energy elements. An analogous strategy is likewise harnessed to affirm the global well-posedness of Schrödinger equations in the $\mathcal{K}^s$ sphere when $\frac{1}{2}\leq s\leq 1$.

Keywords: Half Wave Schrödinger equations, Global Well-Posedness, p Order Nonlinearity.

1. Introduction

The half-wave Schrodinger equation is a time-independent partial differential equation describing a quantum mechanical system. It is generally related to the standard Schrodinger equation but differs in certain aspects. The half-wave Schrodinger equation is a nonlinear equation that is widely used in the field of quantum physics. Precisely, this can portray the transmission of photonic oscillations within non-linear mediums. The non-linear component can function to delineate the alterations in the refractive index which are dependent on variations in light intensity. When dealing with the dynamics of Bose-Einstein condensation (BEC), the half-wave Schrödinger equation can be used to describe the behavior of the system, especially when When considering particle interactions. In quantum fluids, the half-wave Schrödinger equation can describe the flow and dynamic behavior of liquids, especially when nonlinear effects are evident.

Through the study and application of the half-wave Schrödinger equation, we can deepen our understanding of quantum mechanics and nonlinear phenomena and promote the development of quantum science and technology.

2. Definition

We examine the ensuing nonlinear half-wave Schrödinger equation of p order established within the plane (Bahri et al., 2021),

$$\begin{aligned} i\,\partial_t u+\left(\partial_x^2-|D_y|\right)u &= \pm|u|^{p-1}u,\ (x,y)\in\mathbb{R}\times\mathbb{R}, \\ u(0,x,y) &= u_0(x,y). \end{aligned} \tag{1.1}$$

$u_0(x,y)$: function of the initial conditions and represents the value of the function $u(x,y,t)$ at each point (x,y) at time $t=0$.

Note: the rank of the nonlinear component p is assumed to adhere to the conventional range of 1 < p ≤ 2 in equation (1.1). We additionally represent the converging equation as (1.1) - and characterize the diverging equation as (1.1)+. We emphasize that this mathematical formulation sustains the following conservations of mass and energy.

$$M(u):=\frac{1}{2}\int_{\mathbb{R}^2}|u(t,x,y)|^2dxdy=\frac{1}{2}\int_{\mathbb{R}^2}|u_0(x,y)|^2dxdy, \tag{1.2}$$

$$H_{\pm}\big(u(t)\big)=H_{\pm}(u_0) \tag{1.3}$$

where

$$H_{\pm}(u(t)):=\frac{1}{2}\int_{\mathbb{R}^2}\left(|\partial_x u(t,x,y)|^2+\left||D_y|^{\frac{1}{2}}u(t,x,y)\right|^2\right)dxdy\pm\frac{1}{p+1}\int_{\mathbb{R}^2}|u(t,x,y)|^{p+1}dxdy$$

We further present the ensuing anisotropic Sobolev space

$$\mathcal{H}^s:=L_x^2H_y^s(\mathbb{R}^2)\cap H_x^1L_y^2(\mathbb{R}^2),$$

where $\mathcal{H}^{\frac{1}{2}}$ is called the energy space.

We are equally captivated by the semi-wave Schrödinger equations occurring on the cylindrical structure denoted as $\mathbb{R}_x\times\mathbb{T}_y$, which may also be referred to as wave guide Schrödinger equations within the cylindrical domain of $\mathbb{R}_x\times\mathbb{T}_y$,

$$\begin{aligned} i\,\partial_t u+\left(\partial_x^2-|D_y|\right)u &= \pm|u|^{p-1}u,\ (x,y)\in\mathbb{R}\times\mathbb{T}, \\ u(0,x,y) &= u_0(x,y) \end{aligned} \tag{1.4}$$

Moreover, note: we assume that the demand for the nonlinear factor p complies with the condition $1<p\le 2$ in expression (1.4). As before, we denote the concentrating equation with notation (1.4) and represent the deconcentrating equation as (1.4)+. We note that equation (1.4) also maintains the conservation principles of mass and energy, similar to equations (1.2) and (1.3), with integrals occurring over the domain $\mathbb{R}_x\times\mathbb{T}_y$.

Notation

Define the Fourier transform on $\mathbb{R}_x$

$$\hat{g}(\xi):=\mathcal{F}_x(g)(\xi)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-ix\xi}g(x)dx.$$

Define the Fourier transform on $\mathbb{R}_y$

$$h_\eta:=\mathcal{F}_y(h)(\eta)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-i\eta y}h(y)dy.$$

Define the Fourier transform on $\mathbb{T}_y$

$$h_N:=\mathcal{F}_y(h)(N)=\frac{1}{2\pi}\int_{\mathbb{T}}e^{-iNy}h(y)dy,\ N\in\mathbb{Z}.$$

Define the full Fourier transform on $\mathbb{R}_x\times\mathbb{R}_y$

$$(\mathcal{F}U)(\xi,\eta)=\frac{1}{2\pi}\int_{\mathbb{R}}\hat{U}(\xi,y)e^{-i\eta y}dy=\hat{U}_\eta(\xi)$$

Define the "Japanese bracket"

$$\langle\xi\rangle := (1+|\xi|^2)^{1/2},\ \forall\xi\in\mathbb{R}$$

3. Preliminaries

3.1. Strichartz Estimate

Strichartz estimation is an important tool for solving partial differential equations. It provides an effective method to explore the regularity and global properties of solutions for studying nonlinear partial differential equations, especially equations like the half-wave Schrödinger equation. This is when high-order nonlinear effects are significant.

Through a fundamental modification of the validation found in the non-homogeneous prolonged Strichartz approximations (Tzvetkov and Visciglia, 2016), we are able to derive the subsequent Strichartz estimation applicable to semi-wave Schrödinger equations situated on $\mathbb{R}^2$.

Lemma 2.1. Let $(q_1,r_1),(q_2,r_2)$ be two pairs of exponents satisfying

$$\frac{2}{q_j}+\frac{1}{r_j}=\frac{1}{2},\ q_j>2,\ j=1,2$$

Let $s\geq 0$ and $u_0\in L_x^2H_y^s, I$ be an interval of $\mathbb{R}$ and let $f\in L_t^{q_1'}L_x^{r_1'}H_y^s(I\times\mathbb{R}^2)$. Then the unique solution $u\in C(I,\mathcal{S}'(\mathbb{R}^2))$ of

$$i\,\partial_t u+\left(\partial_x^2-\left|D_y\right|\right)u=f, u(0)=u_0 \tag{2.1}$$

belongs to $C\left(I, L_x^2H_y^s(\mathbb{R}^2)\right)\cap L_t^{q_2}L_x^{r_2}H_y^s(I\times\mathbb{R}^2)$, with the estimate

$$\sup_{t\in I}\parallel u(t)\parallel_{L_x^2H_y^s(\mathbb{R}^2)}+\parallel u\parallel_{L_t^{q_2}\left(I;L_x^{r_2}H_y^s(\mathbb{R}^2)\right)}\leq C\left(\|u_0\|_{L_x^2H_y^s(\mathbb{R}^2)}+\parallel f\parallel_{L_t^{q_1'}\left(I;L_x^{r_1'}H_y^s(\mathbb{R}^2)\right)}\right) \tag{2.2}$$

3.2. Brezis-Gallouët Type Inequality

A core mechanism throughout the validation of Theorem 3.1 will be the subsequent discrepancy, akin to the irregularity (Brezis and T. Gallouët, 1979).

Lemma 2.4. Given $s>\frac{1}{2}$, there exists $C_s>0$ such that, $\forall v\in H^s(\mathbb{R})$,

$$\left\|v_\eta\right\|_{L_\eta^1L_x^2(\mathbb{R}^2)}\leq C_s\parallel v\parallel_{L_x^2H_y^{\frac{1}{2}}(\mathbb{R}^2)}\left[\log\left(1+\frac{\parallel v\parallel_{L_x^2H_y^s(\mathbb{R}^2)}}{\parallel v\parallel_{L_x^2H_y^{\frac{1}{2}}(\mathbb{R}^2)}}\right)\right]^{\frac{1}{2}} \tag{2.3}$$

Remark 1. the Brezis-Gallouët type inequality on $\mathbb{R}_x\times\mathbb{T}_y$

$$\|v_N\|_{\ell_N^1L_x^2(\mathbb{R})}\leq C_s\parallel v\parallel_{L_x^2H_y^{\frac{1}{2}}(\mathbb{R}\times\mathbb{T})}\left[\log\left(1+\frac{\parallel v\parallel_{L_x^2H_y^s(\mathbb{R}\times\mathbb{T})}}{\parallel v\parallel_{L_x^2H_y^{\frac{1}{2}}(\mathbb{R}\times\mathbb{T})}}\right)\right]^{\frac{1}{2}} \tag{2.4}$$

See details proof in the appendix Proof 1.

Remark 2. For 2D Brezis-Gallouët inequality: Given $s>1$, there exists $C_s>0$ such that

$$\left\|\hat{v}_{\eta}(\xi)\right\|_{L^{1}_{\xi,\eta}(\mathbb{R}^2)} \leq C_s \parallel v \parallel_{H^1(\mathbb{R}^2)} \left[\log\left(1+\frac{\parallel v \parallel_{H^s(\mathbb{R}^2)}}{\parallel v \parallel_{H^1(\mathbb{R}^2)}}\right)\right]^{\frac{1}{2}}, \ \forall v \in H^s(\mathbb{R}^2) \tag{2.5}$$

Note: the proof of (2.5) will be used in the proof of Corollary 3.2.

3.3. Trudinger Type Estimate

As documented in the findings (Grellier and Gerard, 2015), P. Gerrard and S. Grellier modified the approach previously outlined in (Chemin and Xu, 1997), to derive the Trudinger-style approximation (2.7) applicable to $H^{\frac{1}{2}}(\mathbb{T})$. We will employ a comparable strategy to deduce the subsequent Trudinger-style approximation for $H^{\frac{1}{2}}(\mathbb{R})$. This particular Trudinger-style approximation for $H^{\frac{1}{2}}(\mathbb{R})$ will play a critical role in affirming the singular nature of the frail solutions pertaining to equation (1.1) in Theorem 3.1.

Lemma 2.7. For $u \in H^{\frac{1}{2}}(\mathbb{R})$, we have

$$\parallel u \parallel_{L^k(\mathbb{R})} \leq C\sqrt{k} \parallel u \parallel_{H^{\frac{1}{2}}(\mathbb{R})}, \ \forall k \in (2,\infty) \tag{2.6}$$

See details in the appendix Proof 2.

Remark 3. The Trudinger type estimate for $H^{\frac{1}{2}}(\mathbb{T})$

$$\parallel u \parallel_{L^k(\mathbb{T})} \leq C\sqrt{k} \parallel u \parallel_{H^{\frac{1}{2}}(\mathbb{T})} \tag{2.7}$$

4. Global Well-Posedness

We turn our focus towards scrutinizing the Cauchy problem represented by equation (1.1), aiming to establish the global well-posedness consequences of equation (1.1). Employing a methodology akin to what has been used before, we further deduce the global well-posedness consequences of equation (1.4), as delineated in the study by Burq et al. in 2005. Initially, we set forth to authenticate the subsequent theorem, which substantiates the universal stability of equation (1.1) within the $\mathcal{H}^s$ domain, where the condition $\frac{1}{2} \leq s \leq 1$ holds true.

Theorem 3.1. Let $\frac{1}{2} \leq s \leq 1$. Given $u_0 \in \mathcal{H}^s := L^2_x H^s_y(\mathbb{R}^2) \cap H^1_x L^2_y(\mathbb{R}^2)$, then there is a global solution $u \in C(\mathbb{R};\mathcal{H}^s)$ to (1.1) with $u(0) = u_0$. Furthermore, for every $T > 0$, $u_0 \in \mathcal{H}^s \mapsto u \in C([-T,T],\mathcal{H}^s)$ is continuous.

Proof. The proof has two parts: Part 1 shows the global well-posedness of (1.1) in $\mathcal{H}^s$ with $\frac{1}{2} < s \leq 1$. Part 2 proves the global well-posedness of (1.1) in $\mathcal{H}^{\frac{1}{2}}$.

Part 1: $\frac{1}{2} < s \leq 1$.
Firstly, We set

$$\begin{aligned} X_T &:= C([-T,T];\mathcal{H}^s), \\ \parallel u \parallel_{X_T} &:= \max_{t\in[-T,T]} \parallel u(t) \parallel_{\mathcal{H}^s}. \end{aligned}$$

We also define

$$\Phi_p(u)(t) := \mathrm{e}^{it(\partial_x^2-|D_y|)}u_0 \pm i\int_0^t \mathrm{e}^{i(t-t')(\partial_x^2-|D_y|)}\left(|u(t')|^{p-1}u(t')\right)dt' \tag{3.1}$$

From Lemma 2.1, we have

$$\begin{aligned}\|\Phi_p(u)\|_{X_T} &\leq C\|u_0\|_{\mathcal{H}^s} + C\||u|^{p-1}u\|_{L_t^{q'}\left([-T,T];L_x^{r'}H_y^s(\mathbb{R}^2)\right)} + C\||u|^{p-1}\partial_x u\|_{L_t^{q'}\left([-T,T];L_x^{r'}L_y^2(\mathbb{R}^2)\right)} \\ &\leq C\|u_0\|_{\mathcal{H}^s} + CT^{\frac{1}{q'}} \| u \|_{X_T}^p,\end{aligned}$$

where

$$\frac{1}{q'} = 1 - \frac{1}{q} = \frac{5-p}{4}, \ \frac{1}{r'} = 1 - \frac{1}{r} = \frac{p}{2}$$

Similarly, we have

$$\|\Phi_p(u) - \Phi_p(\tilde{u})\|_{X_T} \leq KT^{\frac{1}{q'}}\max\left(\| u \|_{X_T}^{p-1}, \| \tilde{u} \|_{X_T}^{p-1}\right) \| u - \tilde{u} \|_{X_T}.$$

Thus, if $R > 0$, we get:

$$R > 2C\|u_0\|_{\mathcal{H}^s}, \max(C,K)T^{\frac{1}{q'}}R^{p-1} \leq \frac{1}{2},$$

then the function Φ_p manifests as a contraction mapping within the confined sphere B_R, characterized by a radius R and originating at the zero point within the space X_T. Consequently, it harbors a solitary stable point within B_R. As a result, a specific time duration $T > 0$ and a regional solution denoted by u, encompassed within the continuous function space $C([-T,T];\mathcal{H}^s)$ aligned with equation (1.1), exists such that the initial condition $u(0) = u_0$ is fulfilled.

Subsequently, our attention shifts towards examining the universal prevalence of the aforementioned regional solution. Indeed, courtesy of the principle of energy conservation, the norm $|u(t)|_{H_x^1 L_y^2(\mathbb{R}^2)}$ retains a uniform boundary as function of time t. Leveraging the preliminary deduction, we establish a rupture criterion occurring at a critical time $T^* > 0$.

$$\| u(\tau) \|_{L_x^2 H_y^s(\mathbb{R}^2)} \overset{\tau \to T^*}{\longrightarrow} \infty \tag{3.2}$$

Again by Lemma 2.1, we have

$$\begin{aligned}&\| u(\tau) \|_{L_x^2 H_y^s(\mathbb{R}^2)} \\ \leq\ & C\|u_0\|_{L_x^2 H_y^s(\mathbb{R}^2)} + C\||u|^{p-1}u\|_{L_t^{q'}\left((0,\tau);L_x^{r'}H_y^s(\mathbb{R}^2)\right)} \\ \leq\ & C + C \left\|\| u \|_{L_y^\infty(\mathbb{R})}^{p-1} \| u \|_{H_y^s(\mathbb{R})}\right\|_{L_t^{q'}\left((0,\tau);L_x^{r'}(\mathbb{R})\right)} \\ \leq\ & C + C \left\|\| u_\eta \|_{L_\eta^1(\mathbb{R})}^{p-1} \| u \|_{H_y^s(\mathbb{R})}\right\|_{L_t^{q'}\left((0,\tau);L_x^{r'}(\mathbb{R})\right)} \\ \leq\ & C + C \left\|\| u_\eta \|_{L_x^2 L_\eta^1(\mathbb{R}^2)}^{p-1} \| u \|_{L_x^2 H_y^s(\mathbb{R}^2)}\right\|_{L_t^{q'}(0,\tau)} \\ \leq\ & C + C \left\|\| u_\eta \|_{L_\eta^1 L_x^2(\mathbb{R}^2)}^{p-1} \| u \|_{L_x^2 H_y^s(\mathbb{R}^2)}\right\|_{L_t^{q'}(0,\tau)}.\end{aligned}$$

Recall the inequality (2.3),

$$\|u_\eta\|_{L_\eta^1 L_x^2(\mathbb{R}^2)} \leq C \| u \|_{L_x^2 H_y^{\frac{1}{2}}(\mathbb{R}^2)} \left[\log\left(1 + \frac{\| u \|_{L_x^2 H_y^s(\mathbb{R}^2)}}{\| u \|_{L_x^2 H_y^{\frac{1}{2}}(\mathbb{R}^2)}}\right)\right]^{\frac{1}{2}}.$$

Set $N(\tau):=\| u(\tau) \|^{q^2}_{L^2_x H^s_y(\mathbb{R}^2)}$. We plug (2.3) into the above estimate, since $\| u(t) \|_{L^2_x H^{\frac{1}{2}}_y(\mathbb{R}^2)}$ is uniformly bounded in t, we have

$$N(t) \leq F(t):= C + C\int_0^t N(t')\left[\log\left(2+N(t')\right)\right]^{\frac{q'(p-1)}{2}} dt'.$$

This inequality can be solved by

$$F'(t) = CN(t)[\log(2+N(t))]^{\frac{q'(p-1)}{2}} \leq C(2+F(t))[\log(2+F(t))]^{\frac{q'(p-1)}{2}},$$

so that

$$\frac{d}{dt}[\log(2+F(t))]^{1-\frac{q'}{2}(p-1)} \leq C.$$

Integrating from 0 to τ, we get

$$[\log(2+N(\tau))]^{1-\frac{q'}{2}(p-1)} \leq [\log(2+F(\tau))]^{1-\frac{q'}{2}(p-1)} \leq [\log(2+C)]^{1-\frac{q'}{2}(p-1)} + C\tau,$$

which implies

$$N(\tau) \leq C\mathrm{e}^{C\tau^{\frac{5-p}{7-3p}}}$$

This illustrates that $N(\tau)$ retains a limited value if τ sustains a bounded range. Utilizing the rupture criterion indicated in (3.2), we can infer the universal prevalence of the regional solution. Moreover, by employing a reasoning pattern similar to that used in the validation of the regional issue, we can affirm that any pair of solutions housed within $C(\mathbb{R},\mathcal{H}^s)$ pertaining to equation (1.1), and aligning at $t=0$, are obliged to align across the entire set of real numbers, consequently substantiating the singular nature of the global solution for equation (1.1) within the space $\mathcal{H}^s\left(\frac{1}{2}<s\leq 1\right)$. In a similar vein, we are capable of demonstrating that the transitional map defined as $u_0 \in \mathcal{H}^s \mapsto u \in C([-T,T],\mathcal{H}^s)$ preserves continuity across any chosen timeframe where $T>0$. This culminates the validation of the universal well-defined solutions for equation (1.1) within the domain $\mathcal{H}^s\left(\frac{1}{2}<s\leq 1\right)$.

Part 2: $s=\frac{1}{2}$

Consider u_0 belonging to the space $\mathcal{H}^{\frac{1}{2}}$. We represent u_0 as a progression (u_0^n) in the function space $\mathcal{H}^s$ where $\frac{1}{2}<s\leq 1$. We then designate the series of solutions $(u_n(t))$ present in $C(\mathbb{R};\mathcal{H}^s)$ that align with the initial series (u_0^n). Owing to the principle of energy conservation, the norm of $u_n(t)$ in $\mathcal{H}^{\frac{1}{2}}$ remains confined within bounds for all values of t in $\mathbb{R}$. Consequently, $\partial_t u_n(t)$ maintains a bounded nature within the space $H^{-1}_{x,y}(\mathbb{R}^2)$. From this, we can infer the existence of a certain subset of $u_n(t)$ that weakly gravitates towards $u(t)$ in the $\mathcal{H}^{\frac{1}{2}}$ space, displaying a local uniformity in relation to t. We persist in referring to this subset using the notation $(u_n(t))$. Invoking the Rellich theorem, we discern that $u_n(t)$ approaches $u(t)$ strongly within the space $L^{\gamma}_{loc}(\mathbb{R}^2)$, applicable for every gamma value in the range $2\leq\gamma<6$. Substantiating that u functions as a weak solution to equation (1.1) is a straightforward process.

Next, we establish the singularity of the frail solution utilizing a strategy initially pioneered by V. I. Yudovich (1963) during his analysis of the 2D Euler equation. Assume $u_1(t)$ and $u_2(t)$ serve as two frail solutions to equation (1.1), characterized by initial conditions $u_1(0)=u_2(0)=u_0$. Taking into account $k>2$ and $\tau>0$, we revisit the subsequent notations:

$$\frac{1}{q'}=1-\frac{1}{q}=\frac{5-p}{4},\ \frac{1}{r'}=1-\frac{1}{r}=\frac{p}{2}$$

Then by Lemma 2.1, we have

$$\begin{aligned}
&\|u_1(\tau)-u_2(\tau)\|_{L^2_{x,y}}\\
\le\ & C\||u_1|^{p-1}u_1-|u_2|^{p-1}u_2\|_{L_t^{q'}\left((0,\tau);L_x^{r'}L_y^2\right)}\\
\le\ & C\|(|u_1|^{p-1}+|u_2|^{p-1})|u_1-u_2|\|_{L_t^{q'}\left((0,\tau);L_x^{r'}L_y^2\right)}\\
\le\ & C\left\|\left(|u_1|^{p-1+\frac{1}{k}}+|u_2|^{p-1+\frac{1}{k}}\right)|u_1-u_2|^{1-\frac{1}{k}}\right\|_{L_t^{q'}\left((0,\tau);L_x^{r'}L_y^2\right)}\\
\le\ & C\left\|\left(\|u_1\|^{p-1+\frac{1}{k}}_{L_y^{2k(p-1)+2}}+\|u_2\|^{p-1+\frac{1}{k}}_{L_y^{2k(p-1)+2}}\right)\left\|u_1-u_2\right\|^{1-\frac{1}{k}}_{L_y^2}\right\|_{L_t^{q'}\left((0,\tau);L_x^{r'}\right)}\\
\le\ & C\left\|\left(\|u_1\|^{p-1+\frac{1}{k}}_{L_x^2L_y^{2k(p-1)+2}}+\|u_2\|^{p-1+\frac{1}{k}}_{L_x^2L_y^{2k(p-1)+2})}\right)\left\|u_1-u_2\right\|^{1-\frac{1}{k}}_{L_{x,y}^2}\right\|_{L_t^{q'}(0,\tau)}\\
\le\ & C\left(\|u_1\|^{p-1+\frac{1}{k}}_{L_t^\infty\left((0,\tau);L_x^2L_y^{2k(p-1)+2}\right)}+\|u_2\|^{p-1+\frac{1}{k}}_{L_t^\infty\left((0,\tau);L_x^2L_y^{2k(p-1)+2}\right)}\right)\|u_1-u_2\|^{1-\frac{1}{k}}_{L_{x,y}^2}\ \|_{L_t^{q'}(0,\tau)}
\end{aligned}$$

Rearranging the above formula, we can get:

$$\|u_1(\tau)-u_2(\tau)\|_{L_{x,y}^2}\le Ck^{\frac{p-1}{2}}\tau^{\frac{1}{kq'}}\|u_1-u_2\|^{1-\frac{1}{k}}_{L_t^{q'}\left((0,\tau);L_x^2L_y^2\right)}$$

Then, set

$$g(\tau):=\int_0^\tau\|u_1-u_2\|^{q'}_{L_{x,y}^2}dt,$$

then the above inequalities imply

$$g'(\tau)\le Ck^{\frac{2(p-1)}{5-p}}\tau^{\frac{5-p}{4k}}(g(\tau))^{1-\frac{1}{k}}$$

thus we get

$$g(\tau)\le\left(Ck^{\frac{3p-7}{5-p}}\tau^{\frac{5-p}{4k}+1}\right)^k$$

The latter part of the inequality diminishes to zero as the value of k escalates indefinitely for any given $\tau>0$, thereby affirming the unique character of the fragile solution.

The next step is to substantiate that the fragile solution u exhibits potent continuity over time within the range of $\mathcal{H}^{\frac{1}{2}}$, and its evolution is significantly influenced by the introductory data u_0. Initially, leveraging the principle of mass preservation coupled with the frail convergence of $u_n(t)$ within $L_{x,y}^2(\mathbb{R}^2)$, it enables us to infer that:

$$\|u(t)\|_{L_{x,y}^2(\mathbb{R}^2)}\le\lim_{n\to\infty}\|u_n(t)\|_{L_{x,y}^2(\mathbb{R}^2)}=\lim_{n\to\infty}\|u_0^n\|_{L_{x,y}^2(\mathbb{R}^2)}=\|u_0\|_{L_{x,y}^2(\mathbb{R}^2)}$$

for any $t\in\mathbb{R}$.

By uniqueness condition, we obtain the converse inequality for any $t\in\mathbb{R}$

$$\|u(t)\|_{L_{x,y}^2(\mathbb{R}^2)}=\lim_{n\to\infty}\|u_n(t)\|_{L_{x,y}^2(\mathbb{R}^2)}=\lim_{n\to\infty}\|u_0^n\|_{L_{x,y}^2(\mathbb{R}^2)}=\|u_0\|_{L_{x,y}^2(\mathbb{R}^2)}\tag{3.3}$$

Drawing upon the evidence presented in equation (3.3) and acknowledging that $\|u_n(t)\|_{L^2_{x,y}(\mathbb{R}^2)} = \|u_0^n\|_{L^2_{x,y}(\mathbb{R}^2)}$, it facilitates the logical conclusion that the convergence of $u(t)$ in $L^2_{x,y}(\mathbb{R}^2)$ retains a consistent local uniformity with respect to t. This further implies a robust convergence of $u_n(t)$ towards $u(t)$ within the spatial confines defined by $L^2_{x,y}(\mathbb{R}^2)$, a phenomenon locally uniform in t. Next, we aim to substantiate the robust time-continuity of u within the domain of $\mathcal{H}^{\frac{1}{2}}$. Given that both $u_n(t)$ and $u(t)$ are confined within the bounds of $\mathcal{H}^{\frac{1}{2}}$, the interpolation process, coupled with the locally uniform convergence of $u_n(t)$ to $u(t)$ in $L^2_{x,y}(\mathbb{R}^2)$, allows us to infer a strong convergence of $u_n(t)$ towards $u(t)$ in $L^{p+1}_{x,y}(\mathbb{R}^2)$, an occurrence exhibiting local uniformity with respect to t. Therefore, considering the weak convergence of $u_n(t)$ to $u(t)$ in $\mathcal{H}^{\frac{1}{2}}$, alongside the robust convergence of $u_n(t)$ to $u(t)$ within the boundaries of $L^{p+1}_{x,y}(\mathbb{R}^2)$, supplemented by the principle of energy preservation, it can be stated that: for any $t \in \mathbb{R}$,

$$\begin{aligned}
\frac{1}{2}\| u(t) \|^2_{\mathcal{H}^{\frac{1}{2}}} \pm \frac{1}{p+1}\| u(t) \|^{p+1}_{L^{p+1}_{x,y}(\mathbb{R}^2)} &\leq \frac{1}{2}\liminf_{n\to\infty}\|u_n(t)\|^2_{\mathcal{H}^{\frac{1}{2}}} \pm \frac{1}{p+1}\lim_{n\to\infty}\|u_n(t)\|^{p+1}_{L^{p+1}_{x,y}(\mathbb{R}^2)} \\
&\leq \lim_{n\to\infty}\left(\frac{1}{2}\|u_n(t)\|^2_{\mathcal{H}^{\frac{1}{2}}} \pm \frac{1}{p+1}\|u_n(t)\|^{p+1}_{L^{p+1}_{x,y}(\mathbb{R}^2)}\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{2}\|u_0^n\|^2_{\mathcal{H}^{\frac{1}{2}}} \pm \frac{1}{p+1}\|u_0^n\|^{p+1}_{L^{p+1}_{x,y}(\mathbb{R}^2)}\right) \\
&= \frac{1}{2}\|u_0\|^2_{\mathcal{H}^{\frac{1}{2}}} \pm \frac{1}{p+1}\|u_0\|^{p+1}_{L^{p+1}_{x,y}(\mathbb{R}^2)}
\end{aligned}$$

Similarly, we can get:

$$\frac{1}{2}\| u(t) \|^2_{\mathcal{H}^{\frac{1}{2}}} \pm \frac{1}{p+1}\| u(t) \|^{p+1}_{L^{p+1}_{x,y}} = \lim_{n\to\infty}\left(\frac{1}{2}\|u_n(t)\|^2_{\mathcal{H}^{\frac{1}{2}}} \pm \frac{1}{p+1}\|u_n(t)\|^{p+1}_{L^{p+1}_{x,y}}\right) \tag{3.4}$$

which manifests consistent behavior with respect to t. Integrating the insights from (3.4) along with the regionally consistent convergence of $u_n(t)$ to $u(t)$ within the spectrum of $L^{p+1}_{x,y}(\mathbb{R}^2)$, it becomes evident that $u_n(t)$ demonstrates a robust convergence towards $u(t)$ in the space $\mathcal{H}^{\frac{1}{2}}$, a pattern that retains regional uniformity in relation to t. This consequently indicates the pronounced continuity of $u(t)$ within the $\mathcal{H}^{\frac{1}{2}}$ framework. A parallel rationale can be applied to authenticate the continuous nature of the flow map. This brings us to the conclusion of the substantiation for the global well-posedness of the scenario depicted in equation (1.1) within the $\mathcal{H}^{\frac{1}{2}}$ space. Extending this, we proceed to deduce the global well-posedness pertaining to the forthcoming quadratic half wave Schrödinger equations.

$$\begin{aligned}
i\,\partial_t u + \left(\partial_x^2 - |D_y|\right)u &= \pm|u|u,\ (x,y) \in \mathbb{R}\times\mathbb{R}, \\
u(0,x,y) &= u_0(x,y)
\end{aligned} \tag{3.5}$$

in the higher order Sobolev space $H^2(\mathbb{R}^2)$.

Corollary 3.2. Given $u_0 \in H^2(\mathbb{R}^2)$, there exists a unique global solution $u \in C(\mathbb{R}; H^2(\mathbb{R}^2))$ to (3.5) with $u(0) = u_0$. Also, for every $T > 0$, the flow map $u_0 \in H^2(\mathbb{R}^2) \mapsto u \in C([-T,T], H^2(\mathbb{R}^2))$ is continuous.

Proof. Firstly, we set

$$Y_T := C([-T,T]; H^2(\mathbb{R}^2))$$

$$\| u \|_{Y_T} := \max_{t\in[-T,T]} \| u(t) \|_{H^2(\mathbb{R}^2)}.$$

We recall the definition

$$\Phi_2(u)(t) := e^{it(\partial_x^2 - |D_y|)} u_0 \pm i \int_0^t e^{i(t-t')(\partial_x^2 - |D_y|)} \big(|u(t')| u(t')\big) dt'.$$

We estimate the Y_T norm of $\Phi_2(u)(t)$, with the Minkowski inequality:

$$\|\Phi_2(u)\|_{Y_T} \le \|u_0\|_{H^2(\mathbb{R}^2)} + CT \| u \|_{Y_T}^2$$

Similarly, we have

$$\|\Phi_2(u) - \Phi_2(\tilde{u})\|_{Y_T} \le KT \max\big(\| u \|_{Y_T}, \| \tilde{u} \|_{Y_T}\big) \| u - \tilde{u} \|_{Y_T}.$$

Thus, if $R > 0$,

$$R > 2\|u_0\|_{H^2(\mathbb{R}^2)}, \max(C, K) TR \le \frac{1}{2},$$

subsequently, the function Φ_2 serves as a contractive mapping within the bounded sphere B_R having a radius R and originating at 0 in the Y_T space, thereby possessing a singular stationary point within B_R. This implies the presence of a certain $T > 0$ and a localized solution u that resides in $C\big([-T, T]; H^2(\mathbb{R}^2)\big)$ pertinent to equation (3.5), where $u(0) = u_0$. Subsequently, we venture to scrutinize the universal emergence of this localized solution. Stemming from the prior computations, we manage to establish a specific criterion for abrupt escalation at a certain positive time T^*.

$$\| u(\tau) \|_{H^2(\mathbb{R}^2)} \overset{\tau \to T^*}{\longrightarrow} \infty \tag{3.6}$$

From Theorem 3.1, we hold the global well-posedness of (3.5) in $H^1(\mathbb{R}^2)$, so we have the boundedness with t of $\| u(t) \|_{H^1(\mathbb{R}^2)}$ in any compact subset of $\mathbb{R}$. For $\tau > 0$, according to the Duhamel's formula and Lemma A.3, we get:

$$\begin{aligned} \| u(\tau) \|_{H^2(\mathbb{R}^2)} &\le \|u_0\|_{H^2(\mathbb{R}^2)} + \int_0^\tau \| |u(t)| u(t) \|_{H^2(\mathbb{R}^2)} \, dt \\ &\le \|u_0\|_{H^2(\mathbb{R}^2)} + C \int_0^\tau \|\hat{u}_\eta(t)(\xi)\|_{L^1_{\xi,\eta}(\mathbb{R}^2)} \| u(t) \|_{H^2(\mathbb{R}^2)} \, dt. \end{aligned}$$

Then, plug the 2D Brezis-Gallouët inequality on $\mathbb{R}^2$(2.5), we have:

$$\| u(\tau) \|_{H^2(\mathbb{R}^2)} \le \|u_0\|_{H^2(\mathbb{R}^2)} + C \int_0^\tau \| u(t) \|_{H^1(\mathbb{R}^2)} \left[\log\left(1 + \frac{\| u(t) \|_{H^2(\mathbb{R}^2)}}{\| u(t) \|_{H^1(\mathbb{R}^2)}}\right)\right]^{\frac{1}{2}} \| u(t) \|_{H^2(\mathbb{R}^2)} \, dt$$

Due to $\| u(t) \|_{H^1(\mathbb{R}^2)}$ is bounded in $[0, \tau]$, we deduce that

$$\| u(\tau) \|_{H^2(\mathbb{R}^2)} \le C e^{C_\tau}.$$

Subsequently, employing equation (3.6), we can deduce the universal presence of the solution. The singularity of this global solution coupled with the sustained flow map is attributed to the contractive logic utilized. The demonstration is thus finalized.

Remark 4. In the context of Theorem 3.1, the parameter s cannot be expanded indefinitely owing to the impossibility of anticipating high degrees of regularity concerning the nonlinear component $|u|^{p-1}u$, where $1 < p \le 2$. Likewise, the functional space $H^2(\mathbb{R}^2)$ fails to advance to $H^3(\mathbb{R}^2)$ within Corollary 3.2 for the identical rationale. Furthermore, diverging from numerous other established wellposedness outcomes, our global well-posedness conclusion denotes the exclusivity of the solution within the realm wherein the preliminary data is situated, a consequence stemming from unqualified uniqueness.

Remark 5. The contraction reasoning utilized in establishing Theorem 3.1 facilitates the substantiation that the transition function $u_0 \mapsto u(t)$ adheres to Lipschitz continuity within finite subsets of $\mathcal{H}^s\left(\frac{1}{2} < s \leq 1\right)$. Subsequently, we direct our attention towards the Cauchy problem delineated in (1.4). Indeed, Observation 3 empowers us to modify the strategy evidenced in the substantiation of Theorem 3.1, fostering the derivation of global well-posedness pertaining to (1.4) within the confines of $\mathcal{K}^s$, where $\frac{1}{2} \leq s \leq 1$.

Theorem 3.2. Assume $\frac{1}{2} \leq s \leq 1$. Given u_0 situated in $\mathcal{K}^s := L_x^2 H_y^s(\mathbb{R} \times \mathbb{T}) \cap H_x^1 L_y^2(\mathbb{R} \times \mathbb{T})$, it follows that a distinctive global solution u exists within $C(\mathbb{R}; \mathcal{K}^s)$ in relation to (1.4), satisfying $u(0) = u_0$. Furthermore, for any $T > 0$, the transformation function $u_0 \in \mathcal{K}^s \mapsto u \in C([-T, T], \mathcal{K}^s)$ maintains continuity.

Remark 6. Corroborating the assertion in Remark 4, we also extrapolate the Lipschitz continuous nature of the transformation function $u_0 \mapsto u(t)$ within bounded subsets of $\mathcal{K}^s\left(\frac{1}{2} < s \leq 1\right)$ as delineated in Proposition 3.2.

5. Conclusion

Within this document, our initial focus was on delineating the terminology and framework pertinent to our discourse. Subsequently, we established the Strichartz estimate applicable to the nonlinear half-wave Schrödinger equations - a critical foundational estimate necessary for validating our principal theorem. Furthermore, we incorporated the Brezis-Gallouët type inequality to secure control characterized by a logarithmic nature. These analytic elements are pivotal in substantiating the core theorem concerning Global Well-Posedness. In the segment on Global Well-Posedness, the central theorem was validated, substantiating the global well-posedness of equation (1.1) within the context of $\mathcal{H}^s$, where $\frac{1}{2} \leq s \leq 1$. Specifically, when $\frac{1}{2} < s \leq 1$, the Strichartz estimate coupled with the Brezis-Gallouet type inequality played a decisive role in demonstrating the global well-posedness of equation (1.1) in $\mathcal{H}^s$. To affirm the global well-posedness of (1.1) in the energy domain $\mathcal{H}^{\frac{1}{2}}$, a traditional methodology was employed to formulate the weak solution within the energy space $\mathcal{H}^{\frac{1}{2}}$, followed by leveraging arguments presented by Yudovich in 1963 to validate the singular nature of the weak solution. The continuity inherent to the weak solution is a derivative of both mass and energy preservation principles. Adopting a methodology akin to the one delineated in Frank and Lenzmann's 2013 work for the establishment of the principal theorem, we further extrapolated the global wellposedness of equation (1.4) within $\mathcal{K}^s$, where $\frac{1}{2} \leq s \leq 1$. Looking ahead, our investigative endeavors stand to facilitate a deeper exploration into the potential applications and interpretations of this theory on a global scale, particularly within realms such as quantum mechanics and nonlinear dynamics, fostering advancements in both scientific and engineering domains.

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