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Matrix used in Input and Output Analysis

Litao Zhang *

Furen International School, Singapore

* Corresponding Author Email: 4498zhanglitao@fis.edu.sg

Abstract. This report focuses on Mathematics associated with Economics. It states that the matrix is used in the input-output Analysis in the economic field using Wassily Leontief's Theory. This report comprises six main parts dedicated to comprehensively explaining the origin of Wassily Leontief's theory, its application, and its limitations and improvements. The second part of this paper briefly introduces matrices, such as definition and simple or complicated matrix Algebra. The third part of this passage describes the introduction of the input and output system and its combination with mathematical calculation to derive the demand curve. The fourth part focuses on the background of Professor Wassily Leontief, the derivation of Wassily Leontief's Theory, and the arithmetic of his theory by illustrating two numerical examples. The fifth part discusses real-world applications, points out a general implementation framework based on the thesis, and makes a more summarizing statement. The sixth part considers some typical limitations of real-world applications of Wassily Leontief's input-output Analysis and provides some handle solutions.

Keywords: Matrix, economic, input-output Analysis, Wassily Leontief.

1. Introduction

In addition, the whole structure of this essay is containing five major sectors. The first major part states mathematics related researching background. Matrices are essential tools in many disciplines and appear widely in academic textbooks and mathematics research papers. Matrices are not only important in theoretical mathematics but also play a crucial role in applied mathematics. Therefore, the topic of this essay is not only related to mathematics but also pays more attention to applied mathematics. In particular, this paper explores the application of matrices in economic analysis. Given that there are many examples of prior research in applied mathematics in academics, this article majorly introduces the use of matrices calculation in economics. In previous academic reports, different authors have referred to the same theory: Wassily Leontief's Input-output Analysis.

The second part of this paper briefly sketch the economic model of Input-output Analysis and its application associated with applied mathematics. Input and output analysis can help people determine the economic balance by figure out the relationship between cost and production. Furthermore, economists can deduce the economic models and formulas by using mathematics.

The third part major introduce the professor Wassily Leontief and his primary theory. Wassily Leontief is a famous economist known for his Input-Output Model (IO). This model is widely used in economics, including large-scale businesses and local governments. Based on Input-output analysis, his research is a primary method of quantitative economics, which prints a general blue map of macroeconomic activity as a system of correlated goods and related input. This analysis involves constructing an output table using the mathematics concept of a Matrix. The heart of this theory is using matrices to construct output matrices and analyze production relationships: how one sector change affects the overall economic system. Generally, this theory allows people to analyze the current economic system and predict future economic tendencies.

Various academic papers have already focused on this theory. Therefore, the forth part is focusing the real-life application of Wassily Leontief Theory associated with other authors' researching contribution and providing further contributions of the real-life cases. with Some authors focused on its mathematical operations, like deriving inverse matrix in this theory; some on its real-life applications in experimental reports, like improving resource allocation and predicting economic growth; and others on its limitations in general reports, like lacking ecological validity. So, this paper integrates those precursors' theories and briefly illustrates the general Wassily Leontief's theory. By combining mathematical foundations and economic contribution, this essay provides some constructive comments; due to the criticism of this theory from the academic community, this essay additionally points out some improvement solutions, which offer more comprehensive solutions for IO.

In addition, the fifth part majorly points out the current existing limitations of Wassily Leontief's theory and those issues' optimal solutions.

2. Matrix

2.1. Definition

Matrix is not only widely used in pure mathematics but also generally applied in plenty of fields of subjects associated with math, such as economics. Standing on the perspective of mathematics, the definition of the matrix is that: A matrix is a rectangular array of elements, usually numbers, organized in rows and columns; it was introduced as a way of representing linear maps between finite-dimensional vector spaces [1-3]. In other words, a matrix makes finite-dimensional vector spaces, such as i, j, and k visualization, combining numerical numbers in different dimensions to a matrix table for people to quickly analyze.

So, any table consisting of data can be defined as a matrix. The number of rows and columns determines the size of the matrix. The horizontal line of the matrix is defined as a row, and the vertical line of the matrix is defined as a column. Then, (i, k) refers to the number at row i and column k.In addition, if a matrix contains three rows and three columns, it is a 3*3 matrix.

2.2. Matrix Algebra

Matrices are generalized numbers. That means that it can undertake arithmetical operations. Starting from the essential operation: addition. Those matrices with an identical number of rows and columns can be considered 'same size matrices', and a new matrix will form under the addition of the same size matrices. The latest number of (i, k) is the sum of two numbers of (i, k) from two same-size matrices. Illustrating it by symbol:

Matrix A + Matrix a = Matrix A+a

$$\begin{pmatrix} A & \cdots & B \\ \vdots & \ddots & \vdots \\ C & \cdots & D \end{pmatrix} + \begin{pmatrix} a & \cdots & b \\ \vdots & \ddots & \vdots \\ c & \cdots & d \end{pmatrix} = \begin{pmatrix} A+a & \cdots & B+b \\ \vdots & \ddots & \vdots \\ C+c & \cdots & D+d \end{pmatrix} \tag{1}$$

The same concept and process of subtraction, illustrated by symbol:

Matrix A - Matrix a = Matrix A-a

$$\begin{pmatrix} A & \cdots & B \\ \vdots & \ddots & \vdots \\ C & \cdots & D \end{pmatrix} - \begin{pmatrix} a & \cdots & b \\ \vdots & \ddots & \vdots \\ c & \cdots & d \end{pmatrix} = \begin{pmatrix} A-a & \cdots & B-b \\ \vdots & \ddots & \vdots \\ C-c & \cdots & D-d \end{pmatrix} \tag{2}$$

Beyond such essential operations, matrix algebra also contains matrix multiplication, which means that two or more matrices can be manipulated together. Illustrating it by symbol:

Matric A * Matrix a = Matrix A*a

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} Aa + Bc & Ab + Bd \\ Ca + Dc & Cb + Db \end{pmatrix} \quad (3)$$

2.3. Law of matrix algebra

In a sense, matrices are generalized numbers because they obey many laws that numbers obey, such as Associative Laws, Commutative Laws for Addition, and Commutative Law for Addition. However, the Commutative law for multiplication is one crucial law that numbers satisfy but matrices do not satisfy [2]. Commutative law for multiplication states that ab = ba, but it is not true that AB = BA for matrices. Furthermore, as the table 1. Shown that matrices algebra also satisfies systems of equations by approaching the row reduction method (also named Gauss - Jordan elimination) to achieve echelon form and get the relationships between variables, illustrating it by numerical example:

Table 1. Example of simple Gauss - Jordan elimination.

System of equations	Row operations	Augmented matrix	The matrix in row in echelon form (also called triangular form)		
x + 2y + z = 0 - y + 2z = 0 2x + 3y + z = 0		$\begin{bmatrix} 1 & 2 & 1 & \vdots 1 & 0 & 0 \\ 0 & -1 & 2 & \vdots 0 & 1 & 0 \\ 2 & 3 & 1 & \vdots 0 & 0 & 1 \end{bmatrix}$	2x + 5z = 0 -y + 2z = 0 -3z = 0	L1 + 2L2 → L1	$\begin{bmatrix} 1 & 0 & 5 & \vdots & 1 & 2 & 0 \\ 0 & -1 & 2 & \vdots & 0 & 1 & 0 \\ 0 & 0 & -3 & \vdots & -2 & -1 & 1 \end{bmatrix}$
x + 2y + z = 0 0 y - y + 2z = 0 -1y - z = 0	L3 - 2L2 → L3	$\begin{bmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & -1 & 2 & \vdots & 0 & 1 & 0 \\ 0 & -1 & -1 & \vdots & -2 & 0 & 1 \end{bmatrix}$	3x = 0 -3y = 0 -3z = 0	3L1 + 5L3 → L1 3L2 +2L3 → L2	$\begin{bmatrix} 3 & 0 & 0 & \vdots & -7 & 1 & 5 \\ 0 & -3 & 0 & \vdots & -4 & 1 & 2 \\ 0 & 0 & -3 & \vdots & -2 & -1 & 1 \end{bmatrix}$
1x + 2y + z = 0 -y + 2z = 0 -3z = 0	L3 - L2 →L3	$\begin{bmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & -1 & 2 & \vdots & 0 & 1 & 0 \\ 0 & 0 & -3 & \vdots & -2 & 0 & 1 \end{bmatrix}$	x = 0 y = 0 z = 0	$\frac{1}{3}$L1 →L1 -$\frac{1}{3}$L2 →L2 -$\frac{1}{3}$L3 → L3	$\begin{bmatrix} -\frac{7}{3} & \frac{1}{3} & \frac{5}{3} \\ \frac{4}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$

In addition, the Gauss - Jordan elimination method can also be applied to find out the inverse matrix associating with a system of equations and a 3*3 unit matrix. The matrix is revertible only if the left block can be reduced to identity matrix I. Illustrating by example:

$$\begin{pmatrix} 0.85 & -0.5 & -0.25 & \vdots 1 & 0 & 0 \\ -0.3 & 0.9 & -0.4 & \vdots 0 & 1 & 0 \\ -0.15 & -0.3 & 0.8 & \vdots 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \vdots 1.975 & 1.564 & 1.399 \\ 0 & 1 & 0 & \vdots 0.988 & 2.115 & 1.366 \\ 0 & 0 & 1 & \vdots 0.741 & 1.086 & 2.025 \end{pmatrix} \quad (4)$$

3. Input-Output Analysis

3.1. Definition

The definition of input-output economics is a vast collection of data describing our economic system and as analytical technique for explaining and predicting its behavior of our economic system [4]. However, it is more suitable to state that it is a quantitative economic model that represents the inter-dependencies between different sectors of a national economy or different regional economies [5]. Quickly, input and output analysis can help people determine the economic balance by figure out the relationship between cost and production.

3.2. Input-output Analysis associated with matrix

It is a method of analysis that takes advantage of the relatively stable pattern of the flow of goods and services among the elements of our economy to bring a much more detailed statistical picture of the system into the range of manipulation by economic theory. People can use this method to predict future economic tendencies or generate better results under artificial controlling conditions. Starting with the market equilibrium condition that supply equals demand: $Xi = ai1*Xi + ai2*X2 +...+ain*Xn$. Write Xi for the gross output of product i, let aij denote the amount of good i needed to produce one unit of good j and let Ci denote consumer demand for produce i. Therefore, every sub-item in the original equation is combined to get a general demand curve: $X = AX + c$, where A is the intermediate factor demand named technology matrix and C is the consumer demand.

4. Wassily Leontief Theory

4.1. Background and general description

The Soviet American Economist whose contributions have had a profound impact on both the fields of mathematics and economics. Wassily Leontief's Theory is a revolutionary understanding of how a change in one economic sector can impact others. His work earned him the prestigious Nobel Memorial Prize in Economic Science in 1971. Furthermore, his remarkable achievement was beyond its value and paved a wider academic road for his students to do more profound research. By using Wassily Leontief's Theory, people can advance economic theories and nurture future tendencies. Wassily Leontief's Theory can be considered as an expansion of the boundaries between knowledge subjects since it is a perfect association of mathematics and economics.

Wassily Leontief's Theory in input and output analysis provides a framework for understanding the complex interactions within an economy and has become a fundamental tool in economics. By studying how changes in one sector RIPPLE through the entire system, economists can make more informed decisions and predict the potential impact of policy changes or economic shocks. Stating the last sentence simply, the whole system of economics, matrix system in mathematics, in other words, can be described as a giant, connected well spider web. If spiders change the webs intentionally or the webs change by unintentional factors, the whole web will be changed serially as a Domino Effect. Stating it more academically, if one sector economic system or one item of the matrix changes, the whole results of the system will also change serially.

Using this analysis, he studied trade flows between the United States and other countries. Furthermore, this analysis leads to Leontief's Paradox, which is contrary to expectations. Through this theory, the United States is known for its comparative advantage in capital-intensive products. This Paradox challenged traditional economic theories and sparked a further investigation into the complexity of international trade.

4.2. Stating Wassily Leontief Theory

It is starting from the demand curve equation: $X = AX + C$. A is the technology matrix, and C is the consumer demand mentioned above. The equation can be rearranged as $(I - A) = C$, where I is the identity matrix, so I - A is also a derived matrix. Undergoing the aim of finding the solution of X, using the principle of Commutative law for multiplication, multiplying the inverse matrix of (I -A) on both sides can help people get the equation by $X = (I - A)^{-1} * C$. Furthermore, people can figure out the relationships between each item from this theory. Noticeably, this corresponds to the requirement that any solution to our economic system produces non-negative amounts of each commodity since the commodity numbers and the relationships between commodities cannot be negative [6-7]. Wassily Leontief researched 81 sectors and divided into six major categories of commodities. As shown in Table 2 and Table 3.

Table 2. Six major categories of USA commodities.

	Sector	Examples
FN	Final nonmetal	Leather goods, furniture, food
FM	Final metal	Household appliances
BM	Basic metal	Mining, machine shop products
BN	Basic nonmetal	Glass, wood, textile, livestock
E	Energy	Coal, petroleum, electricity, gas
S	Services	Govt, services, transportation

Table 3. Datasets of six sectors.

	FN	FM	BM	BN	E	S
FN	0.170	0.004	0.000	0.029	0.000	0.008
FM	0.003	0.295	0.018	0.002	0.004	0.016
BM	0.025	0.173	0.460	0.007	0.011	0.007
BN	0.348	0.037	0.021	0.403	0.011	0.048
E	0.007	0.001	0.039	0.025	0.358	0.025
S	0.120	0.074	0.104	0.123	0.173	0.234

However, using the datasets to demonstrate this theory is too complicated. Therefore, using simplified imagery can easily show the theory (Table 4). Illustrating by numerical examples for two scenarios:

The first scenario aims to find out the relationships between commodities.

Table 4. Example of the first scenarios in a virtual economic system.

Coal	Electric	Steel	Purchased by
0.0	0.4	0.6	Coal
0.0	0.1	0.2	Electric
0.4	0.5	0.2	Steel

Deriving matrix a table from the output table

$$\begin{pmatrix} 0.0 & 0.4 & 0.6 \\ 0.6 & 0.1 & 0.2 \\ 0.4 & 0.5 & 0.2 \end{pmatrix} \tag{5}$$

Where the column is the distribution of output and row is needed y sectors

$$\begin{cases} -1.0Pc + 0.4Pe + 0.6Ps = 0 \\ -0.6Pc + 0.9Pe + 0.2Ps = 0 \\ 0.4Pc + 0.5Pe - 0.8Ps = 0 \end{cases} \tag{6}$$

Rearranging it to get sketchy relationships

$$\begin{cases} Pc = \quad + 0.4Pe + 0.6Ps \\ Pe = 0.6Pc + 0.1Pe + 0.2\,Ps \\ Ps = 0.4\,Pc + 0.5Pe + 0.2\,Ps \end{cases} \tag{7}$$

Using Homogeneous equation put into matrix

$$\begin{bmatrix} 1.0 & -0.4 & -0.6 & 0 \\ -0.6 & 0.9 & -0.2 & 0 \\ -0.4 & -0.5 & 0.8 & 0 \end{bmatrix} \tag{8}$$

Using system of equation (mentioned above)

$$\begin{bmatrix} 1.0 & 0.0 & -0.94 & 0.0 \\ 0.0 & 1.0 & -0.85 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \tag{9}$$

Easily get relationship between two commodities

$$\begin{cases} Pc = 0.94\ Ps \\ Pe = 0.85\ Ps \\ Ps = \text{free} \end{cases} \tag{10}$$

The second scenario aims to find out the solutions of X

From Gaussian Elimination (mentioned above) to get

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1.975 & 1.564 & 1.399 \\ 0 & 1 & 0 & 0.988 & 2.115 & 1.366 \\ 0 & 0 & 1 & 0.741 & 1.086 & 2.025 \end{array}\right) \tag{11}$$

Then, $(I - A)^{-1}$ is:

$$\left(\begin{array}{ccc|ccc} 0.85 & -0.5 & -0.25 & 1 & 0 & 0 \\ -0.3 & 0.9 & -0.4 & 0 & 1 & 0 \\ -0.15 & -0.3 & 0.8 & 0 & 0 & 1 \end{array}\right) \tag{12}$$

$(I - A)^{-1} * C$ Is:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1.975 & 1.564 & 1.399 \\ 0 & 1 & 0 & 0.988 & 2.115 & 1.366 \\ 0 & 0 & 1 & 0.741 & 1.086 & 2.025 \end{array}\right) \times \begin{pmatrix} 20 \\ 20 \\ 10 \end{pmatrix} \tag{13}$$

Finally get solution of X:

$$\begin{pmatrix} 79.01 \\ 79.51 \\ 69.93 \end{pmatrix} \tag{14}$$

5. Application

5.1. General

Combining Wassily Leontief's Theory with real-world application, there is a general framework for businesses, companies, or governments to follow. The first part of this framework defines the types of economic systems and categorizes the input-output datasets. The second part of this framework is constructing the output tables, finding a general technology matrix, and using Wassily Leontief's Theory to calculate each sector's inverse matrix and relationships. Then, the third part is making a scenario analysis advocating economic systems to apply the theory to generating situational and more suitable consequences and monitoring, updating the theories into the most suitable one. Next, the fourth step is using the results to analyze economic systems and reflect changes to reach the best outcomes.

5.2. Regional Economic Planning

Designing regional economic planning is associated with Wassily Leontief's Input-Output Theory, IO in the short term and using IO to analyze and optimize the economic events in specific regions. IO is also used to understand the interdependence between different sectors of an economic system. This theory not only enables people to to better understand inter-relationships between each economic sector in the regional economic system but also provides people with general framework to design economic strategies. Utilizing Mary E. Edwards' "Regional and Urban Economics and Economic Development: Theory and Methods," which focuses on Matrix, mathematics related technique, to make input-output analysis and build an output table, aiming to do spatial analysis. People can establish relatively a well-developed economic system [6].

In building well-supported regional economic planning, people can first identify the regional economy's structure, categorizing input and output into different sub-sectors. People can identify the structure of the regional economy by referring to similar methods from Wassily Leontief. This method requires categorizing input-output items into various sub-sectors, which can be derived from Edwards' discussion on regional economic structures. Secondly, gathering the detailed input-output amounts of information from primary regional economic sectors driving growth, people can begin to develop diverse and comprehensive output tables. This table serves as the primary connection between products and services between economic sectors and represents the inter-relationships between the sectors of the regional economic system. Constructing the input-output table and deducing the general output matrix, people can apply Wassily Leontief's IO, as this theory can reveal how changes in demand side impact total output across sectors. Edwards' methodology emphasizes the importance of these relationships, as they are crucial for predicting the outcomes of different economic scenarios. Economists can forecast potential challenges and make more informed decisions by analyzing the shift of the demand curve on the demand side.

Thirdly, applying Wassily Leontief Theory to show how final demand changes affect each sector's total output. Next, make a scenario analysis by simulating various economic policies or external shocks to see how they affect the regional economy. Scenario analysis, a method supported by Edwards, can help economists to assess the potential impacts on the regional economy. This analysis can identify relatively weaker and less competitive sectors and cam highlight those drive economic resilience sectors. Furthermore, based on the analysis, identify strategic sectors that should be prioritized for development to maximize economic growth and sustainability. After these processes, people can advocate for local businesses, community groups and governments to ensure that the consequences reflect local needs and are well combined into the local economic system.

Finally, integrating IO with sustainable development projects to reach a deeper, longer, and more profound future. This method is associated with the first scenario: accessing the relationship between each sector by following the steps outlines — identifying economics structures, constructing output table, conducting scenarios analysis and prioritizing for development to maximize economic economic growth. Edwards' work offers some complex suggestions, offering the methodological foundation and navigating the complexities of regional economic development. This method is associated with the first scenario: accessing the relationship between each sector.

5.3. Industry-Specific Studies

Designing industry-specific studies takes an innovation turn with applying of Wassily Leontief's Input-Output Theory. Ronald E. Miller introduced a groundbreaking methodology that provided a new idea for the design industry by leveraging Wassily Leontief's theory [7]. This method focuses on gathering datasets, building output tables, analyzing economic interactions, and optimizing industry performance.

Firstly, this involves a focused analysis of an industry's economic interactions, like understanding the industry's economic impact, evaluating supply chain dependencies, and assessing the effects of policy changes. Miller highlighted the crucial role of understanding an industry's economic impact, which focused on evaluating supply and output dependencies and policy changes. For example, in a manufacturing industry, economists should focus on supply dependencies to highlight the weaker producing sectors by analyzing the input-output relationships and, based on these results, generate some improvements, aiming to improve the industry resilience.

Secondly, gathering input and output dates. This step is similar to the second step of building regional economics. In addition, Miller emphasized that gathering accurate and comprehensive data collection is crucial. Since the quality and accuracy of Input-output Analysis are heavily based on such data, economists can build and ensure the correct output data table and matrix. The third step, reflects the economic relationships of the specific industry being researched. The third step also involves disaggregating a broader table and exploring the general matrix. Economists can isolate the economic sectors in microcosms and make more relevant and precise analyses. Economists can then create the output table, figure out the general matrix, and calculate the inverse matrix using the Wassily Leontief Theory to see the economic impacts and then analyze the scenario (mentioned in regional economic planning) with strategic interventions and supply chain optimizations. Therefore, economists can quickly discover the relationships of each economic sector in the industry system.

Furthermore, it advocates for industries to apply this theory to stimulate, prepare a comprehensive report, and establish a better industry system.

In conclusion, this application of Wassily Leontief's input-output theory, associated with Miller's viewpoints, helps industry apply this theory to stimulate economic activity and provides comprehensive instructions for economists to evaluate economic activity and make informed decisions.

6. Conclusion

Wassily Leontief only used static datasets in his model. This behaviour limits its ability to capture dynamic economic changes over time since the model relies on fixed coefficients, which assume the constant relationships between each sector in the economic system. It fails to account for technological advancement (improving output level), consumer preferences (changes in supply side), and other economic developments. All these factors can lead to outdated or inaccurate results when applying the model to predict economic tendencies in the future. As a result, the model may only partially represent the complexities of current economic systems that are modern and dynamic.

To solve this problem, people should pay more attention to dynamic input-output analysis. Furthermore, combining the model with complementary approaches, like the marginal utility model, can provide a more comprehensive view of the economy's dynamic nature.

In Wassily Leontief's input-output analysis, it takes a lot of work to retrieve accurate samples. First, for those durable goods, the economic benefits associated with sink cost are long-term, so it is hard to determine its economic sectors and benefits simply in the short term. Second, the sample may change quickly due to the rapid development of technologies. In other words, the results may be invalid since they will be outdated. The third one is that those competitive companies are reluctant to share and publicize their private data. The fourth is the time and workforce costing process to gather big volume datasets.

To solve these problems, people can use more frequent data updates and advanced statistical models to estimate the value of durable goods. Furthermore, people can gather the giant volumes by computing them, like using Python code to cope with them.

References

[1] Serge Lang. Linear Algebra Third Edition, chapter II, 2.1 the space of Matrix

[2] Wang Cheng, Zhang Fusheng. Improvement and application of Domestic demand measurement method: An analysis based on multi-regional input-output model [J]. Contemporary Economy, 2018, 41(08):32-38.

[3] Carl P Simon and Lawrence Blume. (1994) Mathematics for Economists

[4] Huang Yuxia, Xie Jianguo, Hu Fei. Input digitization and domestic value added of manufacturing exports: Empirical analysis based on world input-output database [J]. Journal of international business (foreign economic and trade university), 2024, (4): 37-54.

[5] Wassily Leontief. INPUT- OUTPUT ANALYSIS

[6] Mary E. Edwards. Regional and Urban Economics and Economic Development

[7] Ronald E. Miller. (1963) Inter-industry Economics.