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Taxation Can Significantly Enhance The Effectiveness of Reward and Punishment Mechanisms in The N-Person Snowdrift Game

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Abstract. We extend the reward and punishment mechanisms in the N-person evolutionary snowdrift game (NSG) model by constructing a tax-based pure punishment NSG model and a tax-based pure reward NSG model. Using replicator dynamics, we established two sets of dynamic equations that describe the evolution of the frequencies of the three strategies. Over longer time scales, both systems exhibit stable states involving two strategies, with the exclusion of the third. Introducing tax factors into the reward and punishment mechanisms significantly increases the proportion of cooperators in the steady state and eliminates defectors, with the time to reach the steady state depending on the tax rate. Moreover, we found that tax-based pure reward strategies are more conducive to fostering cooperative behavior within the system compared to tax-based pure punishment strategies. We employed numerical algorithms to simulate the replicator dynamics, and the results of the dynamic equations were in perfect agreement with the numerical simulations.

Keywords: Cooperation; Evolutionary game theory; NSG; reward; punishment.

1. Introduction

Cooperative behavior is highly valued in nature [1-12], but it is easily undermined by individuals who prioritize short-term gains. Without any supporting mechanisms, it is no surprise that cooperation would eventually collapse under the pressure of natural selection [13-19]. Over the past few decades, understanding how cooperation emerges and is sustained among selfish individuals has posed a significant challenge, attracting growing attention across diverse fields, including mathematics, biology, statistical physics, and behavioral science [20-35]. Most empirical research has focused on the public goods game (PGG), a classic multiplayer prisoner's dilemma scenario [36, 37], demonstrating that reward and punishment are powerful tools in overcoming this challenge. The promise of reward or the threat of punishment can effectively induce cooperation among self-interested individuals [38-45].

Inspired by the Public Goods Game (PGG), some scholars have proposed an N-person Snowdrift Game (NSG) [46], in which the payoffs cannot be decomposed into pairwise two-person games. In the NSG, a group of N players, temporarily selected from a larger population, participates in a round of the game. Each player can choose between two actions: cooperation or defection. Typically, the group must complete a task that requires a total cost or effort of c. When the task is completed, every group member receives a benefit r, where $0 < c < r$. At least one cooperative player is needed to accomplish the task. The cost c is shared equally among all cooperators in the group, so the more cooperators there are, the lower the cost per cooperator. Upon task completion, defectors always receive a higher payoff than cooperators. However, if all members choose to defect, they all receive nothing. In an infinitely well-mixed population, cooperators and defectors coexist [46]when the ratio $c/r < 1$ under replicator dynamics. The NSG model has attracted considerable attention. Souza et al. [47] extended the NSG by introducing a minimum threshold of cooperators required to complete the task in the N-person group, revealing changes in the dynamic properties. An in-depth discussion of the NSG can be found in the review by van Veelen and Nowak [48].

Punishment is a common method for regulating behavior. In daily life, people are often willing to pay, for example through taxes, to establish infrastructure like law enforcement and prisons to punish offenders. In the Snowdrift Game (SG) model, various studies have demonstrated that different punishment strategies can promote the emergence of cooperative behavior to varying degrees [49, 50]. Xu introduced punishment strategies into the N-person Snowdrift Game (NSG), developing a three-strategy model involving cooperators, punishers, and defectors [51]. The results indicated coexistence between cooperators and punishers, though the proportion of cooperators within the group remained low. Li introduced a reward mechanism into the NSG, which resulted in a stable coexistence of cooperators and defectors [52], but the proportion of cooperators was still not significant. Currently, research on reward and punishment mechanisms in the NSG is limited. Given the recent advances and the importance of the NSG, there is an urgent need to identify reward and punishment mechanisms that can significantly increase the proportion of cooperators in the NSG.

Recently, Wang [53] introduced a pure punishment (pure reward) strategy incorporating public taxation. In this approach, before the game begins, all participants contribute a tax to a public pool, which is then used to subsidize punishers (rewarders). During the game, punishers (rewarders) incur a cost to penalize defectors (reward cooperators). This tax-based pure punishment (pure reward) mechanism has been shown to significantly increase the proportion of cooperators in Public Goods Games (PGG). However, in Wang’s model, punishers (rewarders) operate independently from cooperators, serving exclusively in their punitive or rewarding roles, which is somewhat disconnected from real-world dynamics. In reality, dedicated institutions for punishment or reward are integral parts of society, contributing to its functioning by collecting taxes to penalize wrongdoers or reward virtuous behavior [54, 55].

Building on this, we introduce a novel tax-based pure punishment (pure reward) mechanism into the N-person Snowdrift Game (NSG) and develop a corresponding NSG model. The evolutionary dynamics within the mixed population are governed by replicator dynamics. After formulating the replicator dynamic equations, we conduct theoretical analysis and numerical simulations to explore the system's evolutionary trends. We illustrate the dynamic flows within the phase space to investigate fixed points and the sensitivity of the system to initial conditions.

2. Model and method

2.1. N-person snowdrift game

In an infinitely large, well-mixed population, a random selection of $N>1$ individuals form a group to participate in a Snowdrift Game. During the game, each participant can choose either to get out of the car and shovel the snow (cooperate) or to stay in the car (defect). The total labor cost of removing the snow is c, and as long as someone chooses to cooperate and shovel, everyone benefits from reaching home, with a benefit of r. For each player, the benefit r of getting home outweighs the cost c of shoveling the snow alone. When there is more than one cooperator, the labor can be shared, reducing the individual cost of cooperation. However, rational individuals are more inclined to maximize their own benefits by free riding on the efforts of others, avoiding the cost altogether. If all participants adopt this strategy, the group's payoff is zero. Furthermore, previous studies [51] have indicated that as the group size N increases, the probability of having at least one cooperator in the group rises, which in turn encourages free-riding behavior among defectors, exacerbating the social dilemma of cooperation.

2.2. Pure punishment mechanism with tax

To overcome the cooperation dilemma described above, we introduce an incentive control strategy within the framework of the snowdrift game, specifically a common tax-based pure punishment strategy. In traditional pure punishment strategies [50], punishers are required to contribute to a punishment pool while engaging in cooperation, resulting in each defector receiving a fine proportional to the contributions (as detailed in the introduction). This often leads to a reduction in the proportion of punishers within an N-person group due to the increased additional costs, making it difficult to diminish the rewards obtained by defectors and thereby reduce their proportion. This paper further incorporates a tax factor into the game model to enhance the effectiveness of the punishment.

All players are mandated to pay a public tax, *K*, into a tax pool. The total tax amount, multiplied by a factor ω_P, is then equally distributed among all punishers within the game group. According to this model, the payoffs for cooperators (C), defectors (D), and punishers with public tax (P) in the N-person snowdrift game can be expressed as:

$$\begin{cases} P_C = r - \dfrac{c}{N_C + N_P + 1} - K \\ p_D = r - \lambda N_P - K \\ P_P = r - M_P - K + \dfrac{\omega_P NK}{N_P + 1} - \dfrac{c}{N_P + N_C + 1} \end{cases} \tag{1}$$

where $N_i (i \in \{C, D, P\})$ denotes the number of individuals using strategy *i* among the N−1 individuals. λN_P represents the fine received by each defector from all the punishers in the group. Specifically, when $N_C + N_P = 0$ is given, there is $r = \lambda = 0$. Notably, when *K*=0, the common tax-based pure punishment strategy is equivalent to the pure punishment strategy. For ease of discussion, ω_P is set to 1 in this paper.

2.3. Pure reward mechanism with tax

Next, we investigate another incentive control strategy, namely the common tax-based pure reward strategy. The traditional pure reward strategy model [52] requires rewarders to contribute an amount M_R to the reward pool while participating in group cooperation, so that each cooperator receives a reward of this amount. However, this approach similarly results in a reduction in the proportion of rewarders in the N-person group due to the increased additional costs, leading to a significant decrease in the proportion of cooperators because they are unable to receive substantial additional rewards. To enhance the incentive intensity of pure rewarders, this paper further incorporates the public tax factor into the pure reward model. All players in the N-person group contribute a fixed tax amount *K* to the tax pool, which is then multiplied by a factor ω_R and equally distributed among all rewarders in the game group. In this case, the payoffs for cooperators (C), defectors (D), and rewarders (R) in the N-person snowdrift game with common tax-based rewards can be expressed as:

$$\begin{cases} P_C = r - \dfrac{c}{N_C + N_P + 1} - K + \gamma N_R \\ P_D = r - K \\ P_R = r - M_R - K + \dfrac{\omega_R NK}{N_R + 1} - \dfrac{c}{N_R + N_C + 1} \end{cases} \tag{2}$$

where $N_j (j \in \{C, D, P\})$ represents the number of individuals using strategy *j* among the N−1 individuals. γN_R denotes the reward received by each cooperator from all rewarders in the group. When $N_C + N_R = 0$, there is $r = \gamma = 0$. Notably, when *K*=0, the common tax-based pure reward strategy is equivalent to the pure reward strategy. For convenience, the value of ω_R is set to 1 in this paper.

3. Theoretical analysis

3.1. Tax-based pure punishment NSG model

The evolutionary dynamics between cooperators, defectors, and punishers with common tax are studied using replicator dynamic equations [56, 57]. Assume that in an N-person group, the proportions of cooperators (C), defectors (D), and punishers (P) are denoted by *x*, *y* and z, respectively, with $x, y, z \in [0,1]$ and $x + y + z = 1$. Let π_C、π_D、π_P represent the expected payoffs of cooperators (C), defectors (D), and punishers (P), respectively. Thus, the expected payoffs can be expressed as follows:

$$\pi_i = \sum_{N_C=0}^{N-1} P_i \binom{N-1}{N_C} \left[\sum_{N_D=0}^{N-N_C-1} \binom{N-N_C-1}{N_D} x^{N_C} y^{N_D} z^{N-N_C-N_D-1} \right] \tag{3}$$

Where $i \in \{C, D, P\}$. $\binom{N-1}{N_C}\binom{N-N_C-1}{N_D} x^{N_C} y^{N_D} z^{N-N_C-N_D-1}$ represents the probability that, among N-1 individuals, the numbers of defectors and punishers are N_D and $N - N_C - N_D - 1$, respectively, given that the number of cooperators is N_C.

Substituting Equation (1) into Equation (3), the expected payoffs for cooperators, defectors, and punishers can be obtained and are respectively expressed as follows:

$$\begin{cases} \pi_C = \dfrac{rcx(N-1)}{N} - \dfrac{N-r}{N} - K \\ \pi_D = \dfrac{rcx(N-1)}{N} - \lambda z(N-1) - K \\ \pi_P = \dfrac{rcx(N-1)}{N} + \dfrac{K[1-(1-z)^N]}{z} - M_P - K \end{cases} \tag{4}$$

The replicator dynamics equations for the N-person group game are given by the following expression:

$$\begin{cases} \dot{x}(t) = x(\pi_C - \overline{\pi}) \\ \dot{y}(t) = y(\pi_D - \overline{\pi}) \\ \dot{z}(t) = z(\pi_P - \overline{\pi}) \end{cases} \tag{5}$$

where $x\pi_C + y\pi_D + z\pi_P$ represents the average payoff of the entire N-person group. Since $x = 1 - y - z$, substituting equation (4) into equation (5) yields:

$$\begin{cases} \dot{y}(t) = yz[M_P - \dfrac{(N-r)c}{N} - \dfrac{K-K(1-z)^N}{z}] - y\ (1-y\ )[\lambda(N-1)z - \dfrac{(N-r)}{N}c] \\ \dot{z}(t) = yz[\lambda(N-1)z - \dfrac{(N-r)c}{N}] - z(1-z)[M_P - \dfrac{(N-r)c}{N} - \dfrac{K-K(1-z)^N}{z}] \end{cases} \tag{6}$$

Here, $(N-1)z$ represents the expected number of punishers among the other N−1 individuals in the N-person game group, $\lambda(N-1)$ represents the expected penalty for each defector, and $(1-z)^N$ represents the probability that no punishers exist in the N-person game group. The existence and stability of equilibrium points in the nonlinear system defined by equation (5) are analyzed theoretically below.

By substituting the variables in equation (5), set $\vartheta(y,z) = \dot{y}(t)$ and $\sigma(y,z) = \dot{z}(t)$ · Thus, finding the equilibrium points of the nonlinear system is equivalent to solving for the common roots of $\vartheta(y,z) = 0$ and $\sigma(y,z) = 0$ · It can be easily computed that in the simplex $Q = \{(x,y,z) \mid x+y+z=1, x,y,z \in [0,1]\}$, the vertices $C(1,0,0)$ · $P(0,0,1)$ · $D(0,1,0)$ are all common roots of $\vartheta(y,z) = 0$ and $\sigma(y,z) = 0$, and therefore, they represent equilibrium points of the nonlinear system. We proceed to solve for the equilibrium points within the interior and along the boundary of the simplex *Q*.

On the boundary CD, wher $z = 0$, it follows that $\vartheta(y,z) = \vartheta(y) = y(1-y)\dfrac{(N-r)c}{N}$ and $\sigma(y,z) = 0$ · Given that $N > r > 0$ in the model, it implies that $\vartheta(y,z) > 0$ is always satisfied. Consequently, there are no equilibrium points on the boundary CD, and the direction of the dynamic evolution is from C to D.

On the boundary PC, where $y = 0$, it follows that $\vartheta(y,z) = 0$ and $\sigma(y,z) = \sigma(z) = z(z-1)[M_P - \dfrac{(N-r)c}{N} - \dfrac{K-K(1-z)^N}{z}]$. Thus, the equilibrium points on boundary PC are the roots for $\sigma(z) = 0$ · Setting $\sigma(z) = 0$ results in $M_P = \dfrac{(N-r)c}{N} + \dfrac{K-K(1-z)^N}{z}$ · Therefore, if there exists a $\varsigma_P \in (0,1)$ such that $M_P = \dfrac{(N-r)c}{N} + \dfrac{K-K(1-\varsigma_P)^N}{\varsigma_P}$, then $(1-\varsigma_P, 0, \varsigma_p)$ is an equilibrium point; otherwise, there are no equilibrium points on boundary PC.

On the boundary PD, where $x = 0$ and $y + z = 1$, it follows that $\vartheta(y,z) = \vartheta(1-z,z)$ and $\sigma(y,z) = \sigma(1\text{-}z,z)$ · Setting $\vartheta(1-z,z) = 0$ and $\sigma(1-z,z)$ yields $M_P = \dfrac{K-K(1-z)^N}{z} + \lambda z(N-1)$. Therefore, if there exists a $\tau_P \in (0,1)$ such that $M_P = \dfrac{K-K(1-\tau_P)^N}{\tau_P} + \lambda\tau_P(N-1)$, then $(0, 1-\tau_P, \tau_P)$ is an equilibrium point; otherwise, there are no equilibrium points on boundary PD.

Inside the simplex $Q=\{(x,y,z)\,|\,x+y+z=1, x,y,z\in[0,1]\}$, setting $\pi_D=\pi_C$ gives $z=\frac{(N-r)c}{\lambda N(N-1)}$, and setting $\pi_P=\pi_C$ gives $M_P=\frac{(N-r)c}{N}+\frac{K-K(1-z)^N}{z}$. If there exist values $\upsilon_P,\psi_P\in(0,1)$ that satisfy Equation (7), then m $(1-\upsilon_P-\psi_P,\psi_P,\upsilon_P)$ is an equilibrium point; otherwise, no equilibrium points exist within the interior of the simplex *Q*.

$$\begin{cases} \upsilon_P+\psi_P\in(0,1) \\ \upsilon_P=\dfrac{(N-r)c}{\lambda N(N-1)} \\ M_P=\dfrac{(N-r)c}{N}+\dfrac{K-K(1-\upsilon_p)^N}{\upsilon_P} \end{cases} \tag{7}$$

Based on the solutions for the equilibrium points, the nonlinear system defined by Equation (6) may have the following equilibrium points: the vertices $C(1,0,0)$, $D(0,1,0)$, and $P(0,0,1)$ of the simplex *Q*; equilibrium points along the boundaries PC and PD; and equilibrium points within the interior of the simplex *Q* We proceed with the stability analysis of these equilibrium points. The Jacobian matrix of the nonlinear system is given by Equation (8):

$$\tilde{A}_P=\begin{bmatrix} \dfrac{\partial\vartheta(y,z)}{\partial y} & \dfrac{\partial\vartheta(y,z)}{\partial z} \\ \dfrac{\partial\sigma(y,z)}{\partial y} & \dfrac{\partial\sigma(y,z)}{\partial z} \end{bmatrix} \tag{8}$$

Setting $\rho_1=\lambda z(N-1)-\frac{(N-r)c}{N}$ and $\rho_2=M_P-\frac{K-K(1-z)^N}{z}-\frac{(N-r)c}{N}$, we obtain:

$$\begin{cases} \dfrac{\partial\vartheta(y,z)}{\partial y}=\rho_1(2y-1)+\rho_2 z+y[(z\dfrac{\partial\rho_2}{\partial y}-(1-y)\dfrac{\partial\rho_1}{\partial y}] \\ \dfrac{\partial\vartheta(y,z)}{\partial z}=y[\rho_2+z\dfrac{\partial\rho_2}{\partial z}-(1-y)\dfrac{\partial\rho_1}{\partial z}] \\ \dfrac{\partial\sigma(y,z)}{\partial y}=z[\rho_1+y\dfrac{\partial\rho_1}{\partial y}-(1-z)\dfrac{\partial\rho_2}{\partial y}] \\ \dfrac{\partial\sigma(y,z)}{\partial z}=\rho_1 y-\rho_2(1-2z)+z[y\dfrac{\partial\rho_1}{\partial z}-(1-z)\dfrac{\partial\rho_2}{\partial z}] \end{cases} \tag{9}$$

By substituting the equilibrium points $C(1,0,0)$ into Equations (8) and (9), we get:

$$\tilde{A}_P = \begin{bmatrix} \frac{(N-r)c}{N} & 0 \\ 0 & \frac{(N-r)c}{N} - M_P + KN \end{bmatrix} \tag{10}$$

since $\frac{(N-r)c}{N} > 0$, the equilibrium point $C(1,0,0)$ is unstable.

By substituting the equilibrium points $D(0,1,0)$ into Equations (8) and (9), we get:

$$\tilde{A}_P = \begin{bmatrix} -\frac{(N-r)c}{N} & M_P - \frac{(N-r)c}{N} - KN \\ 0 & KN - M_P \end{bmatrix} \tag{11}$$

when $KN < M_P$, the equilibrium point $D(0,1,0)$ is stable; $KN \geq M_P$, the equilibrium point $D(0,1,0)$ is unstable.

By substituting the equilibrium points $P(0,0,1)$ into Equations (8) and (9), we get:

$$\tilde{A}_P = \begin{bmatrix} M_P - K - \lambda(N-1) & 0 \\ \lambda(N-1) - \frac{(N-r)c}{N} & M_P - \frac{(N-r)c}{N} - K \end{bmatrix} \tag{12}$$

set $\Omega = \min\{\lambda(N-1), \frac{(N-r)c}{N}\} + K$. If $M_P < \Omega$, then $P(0,0,1)$ is stable; if $M_P \geq \Omega$, then $P(0,0,1)$ is unstable.

By substituting the equilibrium points $(0, 1-\tau_P, \tau_P)$ into Equations (8) and (9), we get:

$$\tilde{A}_P = \begin{bmatrix} (1-\tau_P)[\lambda(N-1)\tau_P - \frac{(N-r)c}{N}] & A_1 \\ \tau_P[\frac{(N-r)c}{N} - \lambda\tau_P(N-1)] & A_2 \end{bmatrix} \tag{13}$$

given $A_1 = (\tau_P - 1)[KN(1-\tau_P)^{N-1} + \frac{(N-r)c}{N} - \frac{1-(1-\tau_P)^N}{\tau_P}]$ and $A_2 = \tau_P\{\lambda(N-1) - \frac{K(\tau_P - 1)[\tau_P N(1-\tau_P)^{N-1} + (1-\tau_P)^N - 1]}{\tau_P^2} - \frac{(N-r)c}{N}\}$. Expanding $(1-\tau_P)^N$ in a Taylor series around $\tau_P = 0$ yields $(1-\tau_P)^N \approx 1 - N\tau_p + o(\tau_P{}^2)$. Substituting this into Equation (13), we obtain:

$$\tilde{A}_P = \begin{bmatrix} M_P - \dfrac{(N-r)c}{N} - KN & 0 \\ \tau_P[\dfrac{(N-r)c}{N} + KN - M_P] & (1-\tau_P)(M_P - KN) \end{bmatrix} \tag{14}$$

given $M_P = \dfrac{K - K(1-\tau_P)^N}{\tau_P} + \lambda\tau_P(N-1) \approx KN + \lambda\tau_P(N-1)$. It is easily verified that $(1-\tau_P)(M_P - KN) > 0$, indicating that the equilibrium point $(0,1-\tau_P,\tau_P)$ is unstable.

Substituting the equilibrium point $(1-\varsigma_P,0,\varsigma_p)$ into Equations (8) and (9) yields:

$$\tilde{A}_P = \begin{bmatrix} \dfrac{(N-r)c}{N} - \lambda\varsigma_P(N-1) & 0 \\ \varsigma_P[\lambda\varsigma_P(N-1) - \dfrac{(N-r)c}{N}] & \dfrac{K(1-\varsigma_P)[\varsigma_P N(1-\varsigma_P)^{N-1} + (1-\varsigma_P)^N - 1]}{\varsigma_P} \end{bmatrix} \tag{15}$$

when $\varsigma_P \geq \dfrac{(N-r)c}{\lambda N(N-1)}$, the equilibrium point $(1-\varsigma_P,0,\varsigma_p)$ is stable; $\varsigma_P < \dfrac{(N-r)c}{\lambda N(N-1)}$, the equilibrium point $(1-\varsigma_P,0,\varsigma_p)$ is unstable.

Substituting the interior equilibrium point $(1-\upsilon_P-\psi_P,\psi_P,\upsilon_P)$ of the simplex *Q* into Equations (8) and (9) yields:

$$\tilde{A}_P = \begin{pmatrix} 0 & \lambda\upsilon_P(\upsilon_P-1)(N-1) \\ 0 & \dfrac{\upsilon_P c(N-r)}{N} \end{pmatrix} \tag{16}$$

given $\upsilon_P = \dfrac{(N-r)c}{\lambda N(N-1)}$. From Equation (16), it follows that the equilibrium point $(1-\upsilon_P-\psi_P,\psi_P,\upsilon_P)$ is unstable.

3.2. Tax-based pure reward NSG model

Assume that in an N-person group, the proportions of cooperators (C), defectors (D), and rewarders (R) are *x*, y and s, respectively, satisfying $x,y,s \in [0,1]$ and $x+y+s=1$. Let π_C、π_D、π_R denote the expected payoffs of cooperators, defectors, and rewarders, respectively. Thus, the expected payoffs can be expressed as:

$$\pi_i = \sum_{N_C=0}^{N-1} P_i \binom{N-1}{N_C} \left[\sum_{N_D=0}^{N-N_C-1} \binom{N-N_C-1}{N_D} x^{N_C} y^{N_D} s^{N-N_C-N_D-1} \right] \tag{17}$$

where $i \in \{C, D, R\}$ · $\binom{N-1}{N_C}\binom{N-N_C-1}{N_D} x^{N_C} y^{N_D} s^{N-N_C-N_D-1}$ represents the probability that, among N−1 individuals, the numbers of defectors and rewarders are N_D and $N-N_C-N_D-1$, respectively, given that the number of cooperators is N_C.

Substituting Equation (2) into Equation (17), the expected payoffs for cooperators, defectors, and rewarders can be obtained and are expressed as follows:

$$\begin{cases} \pi_C = \dfrac{rcx(N-1)}{N} - \dfrac{N-r}{N} + \gamma s(N-1) - K \\ \pi_D = \dfrac{rcx(N-1)}{N} - K \\ \pi_R = \dfrac{rcx(N-1)}{N} + \dfrac{K[1-(1-s)^N]}{s} - M_R - K \end{cases} \tag{18}$$

The replicator dynamics equations for the N-person group game are given by:

$$\begin{cases} \dot{x}(t) = x(\pi_C - \overline{\pi}) \\ \dot{y}(t) = y(\pi_D - \overline{\pi}) \\ \dot{s}(t) = s(\pi_R - \overline{\pi}) \end{cases} \tag{19}$$

where $\overline{\pi} = x\pi_C + y\pi_D + s\pi_R$ represents the average payoff of the entire N-person group. Since $x = 1 - y - s$, substituting Equation (18) into Equation (19) yields:

$$\begin{cases} \dot{y}(t) = ys[M_R - \dfrac{(N-r)c}{N} - \dfrac{K-K(1-s)^N}{s} + \gamma s(N-1)] - y\ (1-y\ )[\gamma(N-1)s - \dfrac{(N-r)}{N}c] \\ \dot{s}(t) = ys[\gamma(N-1)s - \dfrac{(N-r)c}{N}] - s(1-s)[M_R - \dfrac{(N-r)c}{N} - \dfrac{K-K(1-s)^N}{s} + \gamma s(N-1)] \end{cases} \tag{20}$$

let $(N-1)s$ denote the expected number of rewarders among the other N−1 individuals in the N-person game group, $\gamma(N-1)$ represent the expected payoff of each cooperator, and $(1-s)^N$ be the probability that no rewarders are present in the N-person game group. We proceed with a theoretical analysis of the existence and stability of equilibrium points in the nonlinear system defined by Equation (20).

Replacing the variables in Equation (20), let $\vartheta'(y,s) = \dot{y}(t)$ and $\sigma'(y,s) = \dot{s}(t)$. Finding the equilibrium points of the linear system is then equivalent to solving the common roots of $\vartheta'(y,s) = 0$ and $\sigma'(y,s) = 0$. It can be easily calculated that in the simplex $Q' = \{(x,y,s) \mid x+y+s=1, x,y,s \in [0,1]\}$, the three vertices $C(1,0,0)$, $D(0,1,0)$ and $R(0,0,1)$ are common roots of $\vartheta'(y,s) = 0$ and $\sigma'(y,s) = 0$, which means they are equilibrium points of the nonlinear system. We proceed to solve for the equilibrium points within the interior and along the boundaries of the simplex Q'.

On the boundary CD, since $s=0$, it follows that $\vartheta'(y,s)=\vartheta'(y)=y(1-y)\frac{(N-r)c}{N}$ and $\sigma'(y,s)=0$; furthermore, in this model, $N>r>0$ implies that $\vartheta'(y,s)>0$ always holds. Therefore, there are no equilibrium points on the boundary CD, and the direction of dynamic evolution is from C to D.

On the boundary RC, with $y=0$, it follows that $\sigma'(y,s)=\sigma'(s)=s(s-1)[M_R-\frac{(N-r)c}{N}-\frac{K-K(1-s)^N}{s}+\gamma s(N-1)]$ and $\vartheta'(y,s)=0$. Thus, the equilibrium points on boundary RC are determined by the roots of $\sigma'(s)=0$. Setting $\sigma'(s)=0$ gives $M_R=\frac{(N-r)c}{N}+\frac{K-K(1-s)^N}{s}-\gamma s(N-1)$. Therefore, if there exists $\varsigma_R\in(0,1)$ such that $M_R=\frac{(N-r)c}{N}+\frac{K-K(1-\varsigma_R)^N}{\varsigma_R}-\gamma\varsigma_R(N-1)$, then $(1-\varsigma_R,0,\varsigma_R)$ is an equilibrium point; otherwise, no equilibrium points exist on boundary RC.

On the boundary RD, with $x=0$ and $y+s=1$, we have $\vartheta'(y,s)=\vartheta'(1-s,s)$ and $\sigma'(y,s)=\sigma(1\text{-s},s)$. Setting both $\vartheta'(1-s,s)=0$ and $\sigma'(1-s,s)=0$ yields $M_R=\frac{K-K(1-s)^N}{s}$. Therefore, if there exists a $\tau_R\in(0,1)$ such that $M_R=\frac{K-K(1-\tau_R)^N}{\tau_R}$, then $(0,1-\tau_R,\tau_R)$ is an equilibrium point; otherwise, no equilibrium points exist on the boundary RD.

Inside the simplex $Q'=\{(x,y,s)\mid x+y+s=1,x,y,s\in[0,1]\}$, set $\pi_D=\pi_C$ to obtain $s=\frac{(N-r)c}{\gamma N(N-1)}$; then, setting $\pi_P=\pi_C$ yields $M_R=KN$. If there exists on $\upsilon_R,\psi_R\in(0,1)$ that satisfies Equation (21):

$$\begin{cases}\upsilon_R+\psi_R\in(0,1)\\ \upsilon_R=\dfrac{(N-r)c}{\gamma N(N-1)}\\ M_R=KN\end{cases}\tag{21}$$

then $(1-\upsilon_R-\psi_R,\psi_R,\upsilon_R)$ is an equilibrium point; otherwise, there are no equilibrium points within the interior of the simplex Q'.

From the equilibrium point analysis above, the nonlinear system defined by Equation (20) may have the following equilibrium points: the three vertices $C(1,0,0)$, $D(0,1,0)$ and $R(0,0,1)$ of the simplex Q'; equilibrium points on the boundaries RC and RD; and an interior equilibrium point within the simplex. We proceed to analyze the stability of these equilibrium points. The Jacobian matrix of the nonlinear system is given by Equation (22):

$$\tilde{A}_R = \begin{bmatrix} \dfrac{\partial \vartheta'(y,s)}{\partial y} & \dfrac{\partial \vartheta'(y,s)}{\partial s} \\ \dfrac{\partial \sigma'(y,s)}{\partial y} & \dfrac{\partial \sigma'(y,s)}{\partial s} \end{bmatrix} \tag{22}$$

set $\rho_1' = \gamma s(N-1) - \frac{(N-r)c}{N}$ and $\rho_2' = M_R - \frac{K - K(1-s)^N}{s} - \frac{(N-r)c}{N} + \gamma s(N-1)$, we obtain:

$$\begin{cases} \dfrac{\partial \vartheta'(y,s)}{\partial y} = \rho_1'(2y-1) + \rho_2' s + y[(s\dfrac{\partial \rho_2'}{\partial y} - (1-y)\dfrac{\partial \rho_1'}{\partial y}] \\ \dfrac{\partial \vartheta'(y,s)}{\partial s} = y[\rho_2' + s\dfrac{\partial \rho_2'}{\partial s} - (1-y)\dfrac{\partial \rho_1'}{\partial s}] \\ \dfrac{\partial \sigma'(y,s)}{\partial y} = s[\rho_1' + y\dfrac{\partial \rho_1'}{\partial y} - (1-s)\dfrac{\partial \rho_2'}{\partial y}] \\ \dfrac{\partial \sigma'(y,s)}{\partial s} = \rho_1' y - \rho_2'(1-2s) + s[y\dfrac{\partial \rho_1'}{\partial s} - (1-s)\dfrac{\partial \rho_2'}{\partial s}] \end{cases} \tag{23}$$

Substituting the equilibrium point $C(1,0,0)$ into Equations (22) and (23), we obtain:

$$\tilde{A}_R = \begin{bmatrix} \dfrac{(N-r)c}{N} & 0 \\ 0 & \dfrac{(N-r)c}{N} - M_R + KN \end{bmatrix} \tag{24}$$

given $\frac{(N-r)c}{N} > 0$, it follows that the equilibrium point $C(1,0,0)$ is unstable.

By substituting the equilibrium point $D(0,1,0)$ into Equations (22) and (23), we get:

$$\tilde{A}_P = \begin{bmatrix} -\dfrac{(N-r)c}{N} & M_R - \dfrac{(N-r)c}{N} - KN \\ 0 & KN - M_R \end{bmatrix} \tag{25}$$

when $KN < M_R$, the equilibrium point $D(0,1,0)$ is stable; when $KN \geq M_R$, the equilibrium point $D(0,1,0)$ is unstable.

Substituting the equilibrium point $P(0,0,1)$ into Equations (22) and (23), we obtain:

$$\tilde{A}_P = \begin{bmatrix} M_R - K & 0 \\ \gamma(N-1) - \frac{(N-r)c}{N} & M_R - \frac{(N-r)c}{N} - K + \gamma(N-1) \end{bmatrix} \tag{26}$$

let $\Omega' = \min\{0, \frac{(N-r)c}{N} - \gamma(N-1)\} + K$, when $M_R < \Omega'$, the equilibrium point $P(0,0,1)$ is stable; when $M_R \geq \Omega'$, the equilibrium point $P(0,0,1)$ is unstable.

By substituting the equilibrium point $(0,1-\tau_R,\tau_R)$ into Equations (22) and (23), we get:

$$\tilde{A}_R = \begin{bmatrix} (1-\tau_R)[\gamma(N-1)\tau_R - \frac{(N-r)c}{N}] & A_1' \\ \tau_R[-\frac{(N-r)c}{N} + \gamma\tau_R(N-1)] & A_2' \end{bmatrix} \tag{27}$$

given $A_1' = (1-s)\{\gamma\tau_R(N-1) - \frac{K[(1-\tau_R)^N + \tau_R N(1-\tau_R)^{N-1} - 1]}{\tau_R} - \frac{(N-r)c}{N}\}$ and $A_2' = s[\gamma\tau_R(N-1) - \frac{(N-r)c}{N}] + \frac{K(1-\tau_R)[(1-\tau_R)^N + \tau_R N(1-\tau_R)^{N-1} - 1]}{\tau_R}$. When $\tau_R \leq \frac{(N-r)c}{\gamma N(N-1)}$, the main diagonal entries of the Jacobian matrix $\tilde{A}_R$ in Equation (26) are all negative, indicating that the equilibrium point $(0,1-\tau_R,\tau_R)$ is stable. Conversely, when $\tau_R > \frac{(N-r)c}{\gamma N(N-1)}$, the equilibrium point aaa is unstable.

Substituting the equilibrium point $(1-\varsigma_R,0,\varsigma_R)$ with $M_R = \frac{(N-r)c}{N} + \frac{K - K(1-s)^N}{s} - \gamma s(N-1)$ into Equations (22) and (23), we obtain:

$$\tilde{A}_R = \begin{bmatrix} \frac{(N-r)c}{N} - \gamma\varsigma_R(N-1) & 0 \\ \varsigma_R[\lambda\varsigma_R(N-1) - \frac{(N-r)c}{N}] & \varsigma_R(\varsigma_R - 1)\{\gamma(N-1) - \frac{K[(1-\varsigma_R)^N + N\varsigma_R(1-\varsigma_R)^{N-1} - 1]}{\varsigma_R}\} \end{bmatrix} \tag{28}$$

when $\varsigma_R \geq \frac{(N-r)c}{\gamma N(N-1)}$, the equilibrium point $(1-\varsigma_R,0,\varsigma_R)$ is stable; when $\varsigma_R < \frac{(N-r)c}{\gamma N(N-1)}$, the equilibrium point $(1-\varsigma_R,0,\varsigma_R)$ is unstable.

Substituting the interior equilibrium point $(1-\upsilon_R-\psi_R,\psi_R,\upsilon_R)$ of the simplex into Equations (22) and (23) yields:

$$\tilde{A}_R = \begin{pmatrix} 0 & \psi_R\{\gamma(N-1)(\psi_R+\upsilon_R-1)-\frac{K[(1-\upsilon_R)^N+N\upsilon_R(1-\upsilon_R)^{N-1}-1]}{\upsilon_R}\} \\ 0 & \upsilon_R(N-1)(\psi_R+\tau_R-1)+\frac{K(1-\upsilon_R)[(1-\upsilon_R)^N+N\upsilon_R(1-\upsilon_R)^{N-1}-1]}{\upsilon_R} \end{pmatrix} \tag{29}$$

Expanding $(1-\upsilon_R)^N$ in a Taylor series around $\tau_P = 0$ gives $(1-\upsilon_R)^N \approx 1-N\upsilon_R+o(\upsilon_R{}^2)$. Substituting this into Equation (27), we obtain:

$$\tilde{A}_R = \begin{bmatrix} 0 & \gamma\psi_R(\upsilon_R+\psi_R-1)(N-1) \\ 0 & \frac{(\upsilon_R+\psi_R-1)(N-r)c}{N} \end{bmatrix} \tag{30}$$

According to the Central Limit Theorem [58], it can be concluded that the equilibrium point $(1-\upsilon_R-\psi_R,\psi_R,\upsilon_R)$ is not stable.

4. Results

In this section, we present numerical calculations to investigate the evolutionary dynamics of cooperators, defectors, and tax-based pure punishers (or rewarders). Additionally, we conduct a detailed comparative analysis of these dynamics.

Figure 1 illustrates the evolutionary dynamics of the three strategies—cooperators, defectors, and punishers—within the simplex *Q* under different levels of public taxation. Specifically, when *K*=0, the system exhibits four equilibrium points: the unstable point P, the unstable point C, the unstable point Q_1 on the boundary PD, and the stable point. The numerical results show that in the absence of the public tax factor (*K*=0), the system evolves towards a state dominated by defectors (point D), suppressing cooperation within the group.However, when the public tax factor is introduced (*K*=0.3), the system also presents four equilibrium points: the unstable point P, the unstable point C, the unstable point D and a stable point $Q_2(1-\varsigma_P,0,\varsigma_P)$ on the boundary PC. Here, $\varsigma_P \in (0,1)$ is determined by the function $M_P = \frac{K-K(1-\tau_P)^N}{\tau_P}+\lambda\tau_P(N-1)$ and satisfies $\varsigma_P \geq \frac{(N-r)c}{\lambda N(N-1)}$. The results indicate that introducing the public tax factor leads the system to evolve into a state where cooperators and punishers coexist, allowing cooperation to be maintained within the group.These numerical findings are consistent with the theoretical predictions, demonstrating that tax-based pure punishment strategies effectively promote and sustain cooperative behavior in the N-person snowdrift game.

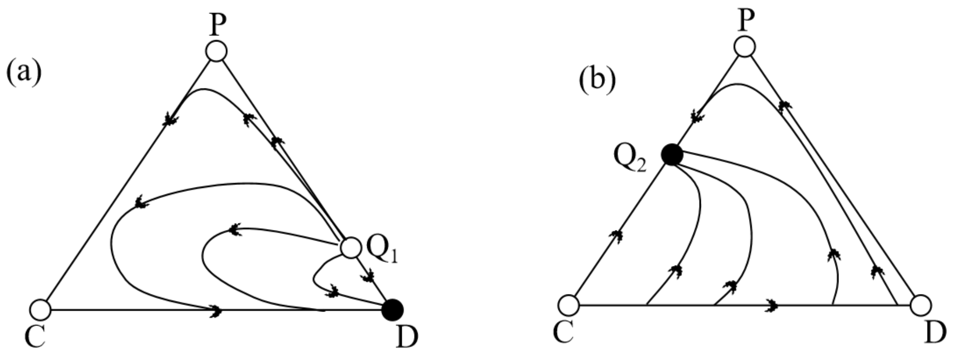

Fig. 1: Evolutionary dynamics of cooperators, defectors, and punishers for different tax values K. The simplex Q represents the system's state space. Points C, D, and P correspond to the three homogeneous states where the entire group consists of cooperators, defectors, and punishers, respectively. In the left panel, there is an equilibrium point Q1 on the boundary PD, while in the right panel, there is an equilibrium point Q2 on the boundary PC. Arrows indicate the direction of system evolution, with hollow points representing unstable equilibria and solid points indicating stable equilibria. The left panel uses a tax value of K=0, and the right panel uses K=0.3. Parameter values: N=6. r=4. c=2. MP=0.8. λ =0.8.

Figure 2 shows the evolutionary dynamics of the three strategies—cooperators, defectors, and rewarders—within the simplex Q' under varying levels of public taxation. Specifically, when *K*=0, the system exhibits three equilibrium points: the unstable point R, the unstable point C, and the stable point D. The numerical results indicate that without the introduction of the public tax factor (*K*=0), the system evolves toward a state dominated by defectors (point D), suppressing cooperation within the group.In contrast, when the public tax factor is introduced (*K*=0.3), the system exhibits five equilibrium points: the unstable points R, C, and D; an unstable point Q_3 on the boundary RD; and a stable point $Q_4(1-\varsigma_R,0,\varsigma_R)$ on the boundary RC The point $Q_4(1-\varsigma_R,0,\varsigma_R)$ is determined by the function $M_R=\frac{(N-r)c}{N}+\frac{K-K(1-\varsigma_R)^N}{\varsigma_R}-\gamma\varsigma_R(N-1)$ and satisfies the $\varsigma_R \ge \frac{(N-r)c}{\gamma N(N-1)}$. The results indicate that introducing public taxation into the strategy allows the system to evolve toward a state where cooperators and rewarders coexist, thereby maintaining group cooperation. Consequently, the numerical results align with the theoretical predictions, demonstrating that tax-based pure reward strategies effectively promote and sustain cooperative behavior in the N-person snowdrift game.

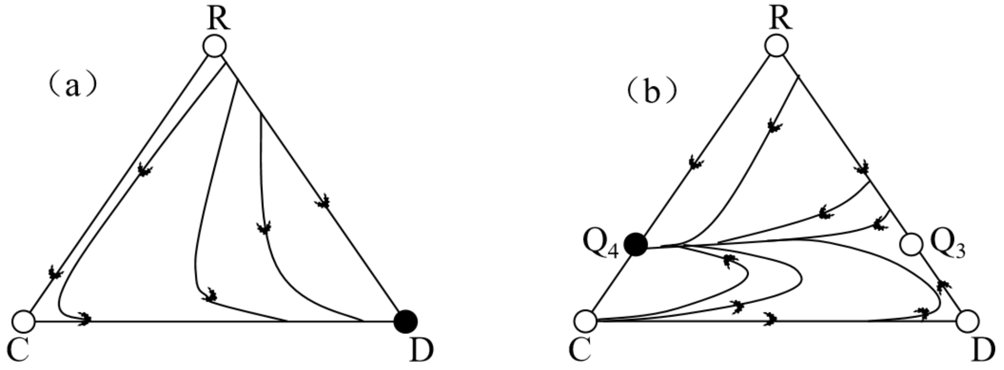

Fig. 2: Evolutionary dynamics of cooperators, defectors, and rewarders for different tax values *K*. The simplex Q' represents the system's state space. Points C, D, and R correspond to the three homogeneous states where the group consists entirely of cooperators, defectors, and rewarders, respectively. In the right panel, an equilibrium point Q_3 is located on the boundary RD, and an equilibrium point Q_4 is found on the boundary RC. Arrows indicate the direction of system evolution, with hollow points representing unstable equilibria and solid points indicating stable equilibria. The left panel sets the tax value at K=0, while the right panel sets it at K=0.3. Parameter values: N=6. r=4. c=2. M_R=0.8. γ=0.8.

Comparing the tax-based pure punishment strategy with the tax-based pure reward strategy reveals that the latter results in a higher level of cooperation. As stated above, if $\varsigma_P \geq \frac{(N-r)c}{\lambda N(N-1)}$ and $\varsigma_R \geq \frac{(N-r)c}{\gamma N(N-1)}$, then both Q_2 and Q_4 are stable points. Theoretical analysis indicates that $\varsigma_P > \varsigma_R$. Additionally, a comparison of Figures 1 and 2 shows that Q_4 is closer to point C than Q_2, suggesting that the system achieves a higher level of cooperation.

5. Conclusion

We introduce public taxation into pure punishment and pure reward strategies, proposing two new NSG model: tax-based pure punishment NSG model and a tax-based pure reward NSG model. The findings indicate that pure punishment (pure reward) strategies are relatively sensitive to public taxation, and both tax-based pure punishment and tax-based pure reward strategies can drive the system toward a steady state where cooperators coexist with punishers (rewarders). This demonstrates that tax-based pure punishment and pure reward strategies are more effective in promoting cooperative behavior and offer an evolutionary advantage over traditional pure punishment (pure reward) strategies.

Furthermore, when comparing the tax-based pure punishment strategy with the tax-based pure reward strategy, the results reveal that the tax-based pure reward strategy leads to a significantly higher proportion of cooperators than the tax-based pure punishment strategy. Thus, the tax-based pure reward strategy is more favorable for the emergence of cooperative behavior compared to the tax-based pure punishment strategy.

Existing literature [51] has shown that under punishment strategies, defectors persist in the system's steady state as group size increases in NSG. This study further confirms this conclusion. However, our research reveals that when the public tax factor is introduced into the pure punishment strategy model, defectors are eliminated from the system's steady state, and free-riders no longer exist, regardless of parameter values.Additionally, this study explores the tax-based pure reward strategy, which has received little attention in previous research. Through theoretical analysis and numerical simulations, the results show that the tax-based pure reward strategy effectively promotes the evolution of cooperation within the group.

Empirical studies on the snowdrift game have demonstrated that punishment strategies are a powerful tool for suppressing free riders within groups and are generally more effective than reward strategies [49]. Interestingly, our theoretical analysis and numerical simulations both indicate that when public taxation is introduced, the pure reward strategy is more effective at fostering cooperative behavior within the group than the pure punishment strategy.

References

[1] M.A. Nowak, Evolving cooperation, J. Theor. Biol. 299 (2012) 1–8.

[2] D.G. Rand, M.A. Nowak, Human cooperation, Trends Cogn. Sci. 17 (2013) 413–425.

[3] M. Perc, J.J. Jordan, D.G. Rand, Z. Wang, S. Boccaletti, A. Szolnoki, Statistical physics of human cooperation, Phys. Rep. 687 (2017) 1–51.

[4] A. Szolnoki, M. Perc, Impact of critical mass on the evolution of cooperation in spatial public goods games, Phys. Rev. E 81 (2010) 057101.

[5] Y. Zhang, F. Fu, X. Chen, G. Xie, L. Wang, Cooperation in group-structured populations with two layers of interactions, Sci. Rep. 5 (2015) 17446.

[6] X. Chen, F. Fu, L. Wang, Influence of different initial distributions on robust cooperation in scale-free networks: a comparative study, Phys. Lett. A 372 (2008) 1161–1167.

[7] Y. Dong, G. Hao, J. Wang, C. Liu, C. Xia, Cooperation in the spatial public goods game with the second-order reputation evaluation, Phys. Lett. A 383 (2019) 1157–1166.

[8] M. Perc, Phase transitions in models of human cooperation, Phys. Lett. A 380 (2016) 2803–2808.

[9] W. Yang, J. Wang, C. Xia, Evolution of cooperation in the spatial public goods game with the third-order reputation evaluation, Phys. Lett. A 383 (2019) 125826.

[10] S. Gao, T. Wu, L. Wang, Evolution of global cooperation and ethnocentrism in group-structured populations, Phys. Lett. A 382 (2018) 2027–2043.

[11] Y. Gong, S. Liu, Y. Bai, Reputation-based co-evolutionary model promotes cooperation in prisoner's dilemma game, Phys. Lett. A 384 (2020) 126233.

[12] X. Wang, W. Chen, J. Zhao, the evolution of cooperation within the multigame environment based on the particle swarm optimization algorithm, Phys. Lett. A 384 (2020) 126165.

[13] M.A. Nowak, Evolutionary Dynamics: Exploring the Equations of Life, Harvard University Press, 2006.

[14] M. Doebeli, C. Hauert, Models of cooperation based on the prisoner's dilemma and the snowdrift game, Ecol. Lett. 8 (2005) 748–766.

[15] G. Szabó, G. Fáth, Evolutionary games on graphs, Phys. Rep. 446 (2007) 97–216.

[16] Z. Wang, S. Kokubo, M. Jusup, J. Tanimoto, Universal scaling for the dilemma strength in evolutionary games, Phys. Life Rev. 14 (2015) 1–30.

[17] J. Tanimoto, H. Sagara, Relationship between dilemma occurrence and the existence of a weakly dominant strategy in a two-player symmetric game, Biosystems 90 (2007) 105–114.

[18] J. Tanimoto, M. Brede, A. Yamauchi, Network reciprocity by coexisting learning and teaching strategies, Phys. Rev. E 85 (2012) 032101.

[19] F. Fu, C.E. Tarnita, N. Christakis, L. Wang, D.G. Rand, M.A. Nowak, Evolution of in-group favoritism, Sci. Rep. 2 (2012) 460.

[20] A. Szolnoki, M. Perc, Reward and cooperation in the spatial public goods game, EPL 92 (2010) 38003.

[21] J.H.W. Tan, F. Bolle, Team competition and the public goods game, Econ. Lett. 96 (2007) 133–139.

[22] F. Fu, T. Wu, L. Wang, Partner switching stabilizes cooperation in coevolutionary prisoner's dilemma, Phys. Rev. E 79 (2009) 036101.

[23] L. Liu, X. Chen, M. Perc, Evolutionary dynamics of cooperation in the public goods game with pool exclusion strategies, Nonlinear Dyn. 97 (2019) 749–766.

[24] A. Szolnoki, M. Perc, Coevolution of teaching activity promotes cooperation, New J. Phys. 10 (2008) 043036.

[25] M. Perc, A. Szolnoki, G. Szabó, Restricted connections among distinguished players support cooperation, Phys. Rev. E 78 (2008) 066101.

[26] J. Tanimoto, Dilemma solving by the coevolution of networks and strategy in a 2 × 2 game, Phys. Rev. E 76 (2007) 021126.

[27] Y. Zhang, F. Fu, T. Wu, G. Xie, L. Wang, Inertia in strategy switching transforms the strategy evolution, Phys. Rev. E 84 (2011) 066103.

[28] X. Chen, A. Szolnoki, Individual wealth-based selection supports cooperation in spatial public goods games, Sci. Rep. 6 (2016) 32802.

[29] K. Li, R. Cong, T. Wu, L. Wang, Social exclusion in finite populations, Phys. Rev. E 91 (2015) 042810.

[30] A. Szolnoki, M. Perc, resolving social dilemmas on evolving random networks, EPL 86 (2009) 30007.

[31] M. Perc, Phase transitions in models of human cooperation, Phys. Lett. A 380 (2016) 2803–2808.

[32] J. Tanimoto, Promotion of cooperation through co-evolution of networks and strategy in a 2 × 2 game, Physica A 388 (2009) 953–960.

[33] J. Tanimoto, Environmental dilemma game to establish a sustainable society dealing with an emergent value system, Physica D 200 (2005) 1–24.

[34] K. Lindgren, M.G. Nordahl, Evolutionary dynamics of spatial games, Physica D 75 (1994) 292–309.

[35] D. Terman, D. Wang, Global competition and local cooperation in a network of neural oscillators, Physica D 81 (1995) 148–176.

[36] J. Tanimoto, Evolutionary Games with Sociophysics: Analysis of Traffic Flow and Epidemics, Springer, 2018.

[37] J. Tanimoto, Difference of reciprocity effect in two coevolutionary models of presumed two-player and multi-player games, Phys. Rev. E 87 (2013) 062136.

[38] M. Milinski, B. Rockenbach, On the interaction of the stick and the carrot in social dilemmas, J. Theor. Biol. 299 (2012) 139–143.

[39] M. Sutter, S. Haigner, M.G. Kocher, Choosing the carrot or the stick? Endogenous institutional choice in social dilemma situations, Rev. Econ. Stud. 77 (2010) 1540–1566.

[40] M. Sefton, R. Shupp, J.M. Walker, The effect of rewards and sanctions in provision of public goods, Econ. Inq. 45 (2007) 671–690.

[41] Y. Wu, S. Chang, Z. Zhang, Z. Deng, Impact of social reward on the evolution of the cooperation behavior in complex networks, Sci. Rep. 7 (2017) 41076.

[42] S. Wang, X. Chen, A. Szolnoki, Exploring optimal institutional incentives for public cooperation, Commun. Nonlinear Sci. Numer. Simul. 79 (2019) 104914.

[43] L. Liu, S. Wang, X. Chen, M. Perc, Evolutionary dynamics in the public goods games with switching between punishment and exclusion, Chaos 28 (2018) 103105.

[44] F. Fu, M.A. Nowak, C. Hauert, Invasion and expansion of cooperators in lattice populations: prisoner’s dilemma vs. snowdrift games, J. Theor. Biol. 266 (2010) 358–366.

[45] Y. Zhang, T. Wu, X. Chen, G. Xie, L. Wang, Mixed strategy under generalized public goods games, J. Theor. Biol. 334 (2013) 52–60.

[46] D.-F. Zheng, H. Yin, C. Chan, P.M. Hui, Europhys. Lett. EPL 80 (2007) 18002.

[47] M.O. Souza, J.M. Pacheco, F.C. Santos, J. Theoret. Biol. 260 (2009) 581.

[48] M. van Veelen, M.A. Nowak, J. Theoret. Biol. 292 (2012) 116.

[49] Gao, Lei, et al. "Collective punishment is more effective than collective reward for promoting cooperation." Scientific Reports 5.X(2015):17752.

[50] Xu, C., et al. "Costly punishment and cooperation in the evolutionary snowdrift game." Physica, A. Statistical mechanics and its applications 390.9(2011).

[51] Xu, Meng, et al. "Cooperative behavior in N-person evolutionary snowdrift games with punishment." Physica, A. Statistical mechanics and its applications 424-(2015).

[52] Li, Kun, et al. "Pool-rewarding in N-person snowdrift game." Chaos Solitons & Fractals 143(2021):110591.

[53] Wang, Shengxian, L. Liu, and X. Chen. "Tax-based pure punishment and reward in the public goods game." Physics Letters A 386(2021):126965.

[54] A. Szolnoki, M. Perc, Europhys. Lett. EPL 92 (2010) 38003.

[55] C. Hilbe, K. Sigmund, Proc. R. Soc. B 277 (2010) 2427.

[56] P. Schuster, K. Sigmund, Replicator dynamics, J. Theor. Biol. 100 (1983) 533–538.

[57] J. Hofbauer, K. Sigmund, Evolutionary game dynamics, Bull. Am. Math. Soc. 40 (2003) 479–519.

[58] Bérard, Jean, P. Del Moral, and A. Doucet. "A Lognormal Central Limit Theorem for Particle Approximations of Normalizing Constants." Electronic Journal of Probability 19(2014):1---28.