

# Governing equation for a rocket going around the sun and returning to the earth

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**Abstract.** In the article, the author mainly focuses on how to launch a rocket from the earth to go around the sun and return to the earth. The equation and solution of this motion are obtained by means of Newtonian mechanics, calculus and conic curve. To find the expression of the trajectory of the rocket, the article will obtain Kepler's Law by using Newton's Law, after setting a polar coordinate system. Similarly, plenty of mathematical methods are used to obtain the result in the article, hence, the combination of mathematic with physics and the application of pure mathematic is the tenet of the article. The most important mathematical method is solving differential equation, which is widely used in not only mechanics, but even quantum mechanics. After obtaining the solution, the author plugs the values into the analytic expression, in order to better reflect the actual role of the research results.

**Keywords:** Applied Mathematics; Analytic Geometry; Classical Mechanics; Rocket Orbit; Astrophysics; Calculus.

## 1. Introduction

Newton's classical mechanics is a very important part of physics, which was established by Lord Newton in 1687. This achievement was symbolized in his book *Principia Mathematica Natural philosophy* which elaborated Newton's three laws of the motion which formed the basis of classical mechanics and which would have a profound influence on later physics and engineering.[1] Newton's contribution not only marked that man's understanding of the laws of nature reached a new height but also exerted an extremely profound impact on the course of scientific development and human's way of production, life and thinking. Astrophysics is a branch of classical mechanics, involving rocket's orbit trajectory in the article.[2] Human space travel is now being hampered; therefore, some practical application of physics need to be paid attention to. For example, Coriolis force applied on the rocket launching and the gravitational distortion of space caused by black holes.[3] In the classical mechanics, Astrophysics is a very important branch, which mainly studies the orbit of celestial bodies.

In a final exam when Qian Xuesen was teaching at Tsinghua, he gave mechanics students a question. The problem is to launch a rocket from the earth around the sun, before returning to the earth, please list the equation and find the solution. After a whole day of thinking, no one of the students of Tsinghua Mechanics made it, and all of them were held back. This article tried to give the equation of the trajectory and solve the value. With these values and the equations, humans can make major breakthrough in jet-powered rockets and perfect the theoretical research that solve rocket orbits.[4] After hundreds of years of development, classical mechanics has been perfected. Thus, applied mathematics and applied physics are main subjects of theoretical study.[5]

## 2. Method and Theory

### 2.1. background knowledge and method

The article solves the differential equation by some methods, please see the author's another essay on solving second order differential equations.

As for the basic knowledge of cross product



$$\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}, \vec{b} = b_x \vec{i} + b_y \vec{j} + b_z \vec{k}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y) \vec{i} + (a_z b_x - a_x b_z) \vec{j} + (a_x b_y - a_y b_x) \vec{k} \quad (1)$$

The auxiliary Angle formula is a very important tool in dealing with the rocket orbit equation.

$$A \sin a + B \cos a = \sqrt{A^2 + B^2} \left( \frac{A}{\sqrt{A^2 + B^2}} \sin a + \frac{B}{\sqrt{A^2 + B^2}} \cos a \right)$$

$$\frac{A}{\sqrt{A^2 + B^2}} = \cos t, \frac{B}{\sqrt{A^2 + B^2}} = \sin t \quad (2)$$

$$A \sin a + B \cos a = \sqrt{A^2 + B^2} \sin(a + t)$$

$$\sin t = \frac{B}{\sqrt{A^2 + B^2}}, \cos t = \frac{A}{\sqrt{A^2 + B^2}}, \tan t = \frac{B}{A}$$

In determining the shape of the rocket orbit, the polar equation of the ellipse is used. If the article assumes  $(-c,0)$  as the origin, then

$$r + \sqrt{(r \cos \theta - 2c)^2 + (r \sin \theta)^2} = 2a$$

$$r = \frac{b^2}{a - c \cos \theta} = \frac{\frac{b^2}{a}}{1 - e \cos \theta} = \frac{ep}{1 - e \cos \theta} \quad (3)$$

## 2.2. Structure and content of the article

In the first part of Sec. 3.1, the author uses Newton's second law and polar coordinates to make the force analysis of the rocket is carried out. According to the point and cross product of the vector, the acceleration vector and the coordinate vector have been calculated. Finally, the author got the simplest form of the derivative of the angle and the polar. [6] In the second Sec. 3.2, by methods of separation of variables, the article has found the solution of the differential equation in Sec. 3.1. Then, by using a substitution and combing a few initial boundary conditions, the author got the expression of polar coordinate, which is amazingly a conic curve. Then in the Sec. 3.3, the authors fit the earth orbit equation with the sun as the origin in order to find the solution of how the rocket would return to the earth. In the Sec. 3.4, the author took the accurate value back to the expression and found the velocities and the period of the rocket.

## 3. Results and Application

### 3.1. Governing equation for rocket trajectories around the sun

The rocket trajectory is assumed to be a plane curve, and a plane rectangular coordinate system is established with the position of the sun as the origin.[7]

Then the rocket's position vector is

$$r(t) = xi + yj = r \cos \theta i + r \sin \theta j \quad (4)$$

The gravitational pull of the rocket on the sun is

$$F = -G \frac{Mm}{\|r\|^2} \frac{r}{\|r\|} \quad (5)$$

Where  $G$ ,  $M$ , are gravitation const, the mass of the sun and the mass of the rocket. According to the Newton's second law,

$$F = -G \frac{Mm}{\|r\|^2} \frac{r}{\|r\|} = m\ddot{r} \quad (6)$$

Take the derivative of the vector  $r$ , the author will find the expression of the velocity vector

$$\dot{r} = (\dot{r} \cos \theta - r \sin \theta \dot{\theta})i + (\dot{r} \sin \theta + r \cos \theta \dot{\theta})j \quad (7)$$

Then, the author can know radial velocity is

$$v = \frac{1}{r}(\dot{r} \cdot r) = \dot{r} \quad (7a)$$

Tangential velocity can be expression as

$$v_h = \dot{r} \cdot \tau = r\dot{\theta} \quad (7b)$$

Where,  $\tau = \frac{1}{r}(k \times r) = -\sin \theta i + \cos \theta j$ , which is tangential unit vector. If the author takes the derivative of vector  $r$ , the acceleration vector of the rocket can be got.

$$\ddot{r} = (\ddot{r} \cos \theta - 2\dot{r}\dot{\theta} \sin \theta - r\dot{\theta}^2 \cos \theta - r \sin \theta \ddot{\theta})i + (\ddot{r} \sin \theta + 2\dot{r}\dot{\theta} \cos \theta - r\dot{\theta}^2 \sin \theta + r \cos \theta \ddot{\theta})j \quad (8)$$

Because vector  $r$  is parallel to the  $F$ , the article can get that

$$\ddot{r} \times r = 0 \quad (9)$$

By simplification, the author will know

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \quad (10)$$

Combing (7) and (5), the article can get

$$\ddot{r} = (\ddot{r} - r\dot{\theta}^2)(\cos \theta i + \sin \theta j) = (\ddot{r} - r\dot{\theta}^2) \frac{r}{\|r\|} \quad (11)$$

Combing (8) and (3), the article will get

$$\ddot{r} - r\dot{\theta}^2 + G \frac{M}{r^2} \quad (12)$$

Hence, the article has gotten the trajectory governing equation, which is

$$\ddot{r} - r\dot{\theta}^2 + G \frac{M}{r^2}, 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \quad (13)$$

### 3.2. The solution to the rocket trajectory

To find the solution of the differential equation in (10), the author tries to use method of separation of variables

$$\frac{2\dot{r}}{r} = -\frac{\ddot{\theta}}{\dot{\theta}}, 2\ln r = -\ln\dot{\theta} + C, r^2 = \frac{\exp(C)}{\dot{\theta}} \quad (14)$$

Amazingly, the article gets an expression

$$\frac{1}{2}\dot{\theta}r^2 = A = \text{const} \quad (15)$$

Which can be considered as, the area of the rocket path swept per unit time is const. This is the classical Kepler's second law. Then, to find the solution of the first differential equation in (10), the author tries to use the substitution

$$r = \frac{1}{u} \quad (16)$$

$$\dot{\theta} = 2Au^2 \quad (17)$$

$$\dot{r} = \frac{dr}{du} \frac{du}{d\theta} \frac{d\theta}{dt} = -u^{-2} \frac{du}{d\theta} 2Au^2 = -2A \frac{du}{d\theta} \quad (18)$$

$$\ddot{r} = \frac{d\dot{r}}{d\theta} \frac{d\theta}{dt} = -2A \frac{d^2u}{d\theta^2} 2Au^2 = -4A^2u^2 \frac{d^2u}{d\theta^2} \quad (19)$$

Combing these results with the differential equation in (10), the author can get

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{4A^2} \quad (20)$$

The solution of the differential equation is

$$u = \frac{GM}{4A^2} + C_2 \sin \theta + C_1 \cos \theta \quad (21)$$

$$\dot{u} = -C_1 \sin \theta + C_2 \cos \theta \quad (22)$$

Because when  $t = 0, \theta = 0$ , taking the value back to (18) and (19), the author can get

$$u(\theta = 0) = \frac{GM}{4A^2} + C_1 \quad (23)$$

$$\dot{u}(\theta = 0) = C_2 \quad (24)$$

Combing (13) and (15), the author can get

$$r(0) = \frac{1}{u(0)} \quad (25)$$

$$\dot{r}(0) = -2A\dot{u}(0) \quad (26)$$

Thus, the author can use method of undetermined coefficients to find the value of the const in the expression

$$C_1 = \frac{1}{r(0)} - \frac{GM}{4A^2} \quad (27)$$

$$C_2 = -\frac{\dot{r}(0)}{2A} \quad (28)$$

Then, the article takes the value back to (18), and can get

$$\begin{aligned} u &= \frac{GM}{4A^2} + \left(\frac{1}{r(0)} - \frac{GM}{4A^2}\right) \cos \theta - \frac{\dot{r}(0)}{2A} \sin \theta \\ &= \frac{GM}{4A^2} + \sqrt{\left(\frac{1}{r(0)} - \frac{GM}{4A^2}\right)^2 + \left(\frac{\dot{r}(0)}{2A}\right)^2} \cos(\theta + \varphi) \\ &= \frac{GM}{4A^2} \left(1 + \frac{4A^2}{GM} \sqrt{\left(\frac{1}{r(0)} - \frac{GM}{4A^2}\right)^2 + \left(\frac{\dot{r}(0)}{2A}\right)^2} \cos(\theta + \varphi)\right) \end{aligned} \quad (29)$$

Were

$$\sin \varphi = \frac{\frac{\dot{r}(0)}{2A}}{\sqrt{\left(\frac{1}{r(0)} - \frac{GM}{4A^2}\right)^2 + \left(\frac{\dot{r}(0)}{2A}\right)^2}}, \quad \cos \varphi = \frac{\frac{1}{r(0)} - \frac{GM}{4A^2}}{\sqrt{\left(\frac{1}{r(0)} - \frac{GM}{4A^2}\right)^2 + \left(\frac{\dot{r}(0)}{2A}\right)^2}}, \quad \tan \varphi = \frac{\frac{\dot{r}(0)}{2A}}{\frac{1}{r(0)} - \frac{GM}{4A^2}} \quad (30)$$

Hence, the article takes all the values of parameters back to the expression of vector r

$$r = \frac{\frac{4A^2}{GM}}{1 + \frac{4A^2}{GM} \sqrt{\left(\frac{1}{r(0)} - \frac{GM}{4A^2}\right)^2 + \left(\frac{\dot{r}(0)}{2A}\right)^2} \cos(\theta + \varphi)} = \frac{ep}{1 + e \cos(\theta + \varphi)} \quad (31)$$

The article amazingly finds that the expression of vector r in (28) is a conic curve, the eccentricity of it is

$$e = \frac{4A^2}{GM} \sqrt{\left(\frac{1}{r(0)} - \frac{GM}{4A^2}\right)^2 + \left(\frac{\dot{r}(0)}{2A}\right)^2} \quad (32)$$

Having analyzed the expression of the eccentricity, the article found that, eccentricity depends on the initial orbital altitude, radial velocity and const A. Now, discuss the const A. [8]

If the author set the altitude of the rocket from the sun's surface when it enters orbit is H, and the radius of the sun is R. The author assumed  $R_o = R + H$ , and the tangential velocity of entry into orbit is  $v_h$

According to the angular momentum conservation, the author can know

$$mv_h R_o = m\dot{\theta} r^2 = 2mA, A = \frac{1}{2} v_h R_o \quad (33)$$

The author can know that A is determined by the horizontal velocity at orbit entry. If the article takes the time of entering orbit as zero, then  $r(0) = R_o, \dot{r}(0) = v$ . Since the rocket is returning to Earth, its orbit should be elliptical. Otherwise, the rocket will escape the solar system and not meet the earth when the orbit is a hyperbola or a parabola. This requires the eccentricity of less than 1.

$$\left(\frac{1}{R_o} - \frac{GM}{(v_h R_o)^2}\right)^2 + \left(\frac{v}{v_h R_o}\right)^2 < \left(\frac{GM}{(v_h R_o)^2}\right)^2 \quad (34)$$

$$\left(1 - \frac{GM}{v_h^2 R_o}\right)^2 + \left(\frac{v}{v_h}\right)^2 < \left(\frac{GM}{v_h^2 R_o}\right)^2 \quad (35)$$

Which is

$$v^2 + v_h^2 < 2v_1^2 = v_2^2 \quad (36)$$

Where,  $v_1 = \sqrt{GM/R_o}, v_2 = \sqrt{2GM/R_o}$ , these are the first and second cosmic velocities of the sun.

Because the rocket goes into orbit at earth,  $R_o$  can be the distance between the sun and the centre of the earth. By looking up the information, the size of the earth's orbit is

$$a = 1.4960 \times 10^8 km, b = 1.4958 \times 10^8 km \quad (37)$$

Taking the average of the two, the author can get

$$R_o = 1.4959 \times 10^8 km \quad (38)$$

Estimating the mass of the sun

$$M = \frac{4\pi^3 R^2}{GT^2} = \frac{4\pi^2 \times (1.496 \times 10^{11})^3}{6.67 \times 10^{-11} \times (365 \times 24 \times 3600)^2} = 1.994 \times 10^{30} kg \quad (39)$$

According to the data calculated above,

$$v_1 = \sqrt{GM/R_o} = \sqrt{\frac{6.67 \times 10^{-11} \times 1.994 \times 10^{30}}{1.4959 \times 10^{11}}} = 29818 m/s \quad (40)$$

$$v_2 = \sqrt{2GM/R_o} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 1.994 \times 10^{30}}{1.4959 \times 10^{11}}} = 42169 m/s \quad (41)$$

Now expressing the rocket's orbit in terms of its entry parameters

$$ep = \frac{4A^2}{GM} = \frac{(v_h R_o)^2}{GM} = \frac{v_h^2}{v_1^2} R_o \quad (42)$$

$$e = \frac{4A^2}{GM} \sqrt{\left(\frac{1}{r(0)} - \frac{GM}{4A^2}\right)^2 + \left(\frac{\dot{r}(0)}{2A}\right)^2} = \frac{v_h^2}{v_1^2} \sqrt{\left(1 - \frac{v_1^2}{v_h^2}\right)^2 + \left(\frac{v}{v_h}\right)^2} \quad (43)$$

$$\sin \varphi = \frac{4A^2}{eGM} \frac{\dot{r}(0)}{2A} = \frac{4A^2}{eGM} \frac{v}{v_h R_o} \quad (44)$$

$$\cos \varphi = \frac{4A^2}{eGM} \left( \frac{1}{r(0)} - \frac{GM}{4A^2} \right) = \frac{4A^2}{eGM R_o} \left( 1 - \frac{v_1^2}{v_h^2} \right) \quad (45)$$

$$\tan \varphi = \frac{\frac{\dot{r}(0)}{2A}}{\frac{1}{r(0)} - \frac{GM}{4A^2}} = \frac{v v_h}{v_h^2 - v_1^2} \quad (46)$$

Taking all the value back to (28)

$$r(\theta) = \frac{ep}{1+e \cos(\theta+\varphi)} = \frac{\frac{v_h^2 R_o}{v_1^2}}{1 + \frac{v_h^2}{v_1^2} \sqrt{\left(1 - \frac{v_1^2}{v_h^2}\right)^2 + \left(\frac{v}{v_h}\right)^2} \cos(\theta+\varphi)} \quad (46a)$$

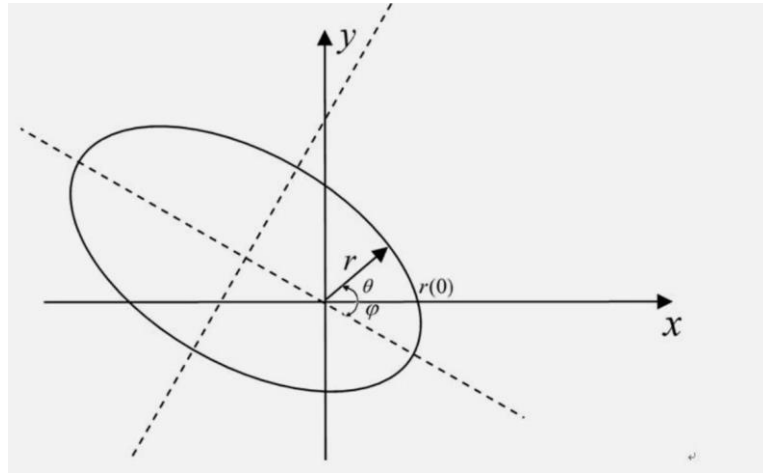
$$\varphi = \tan^{-1} \frac{v v_h}{v_h^2 - v_1^2}, \text{ for } |v_h| > v_1, \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad (46b)$$

$$\varphi = \frac{\pi}{2}, \text{ for } |v_h| = v_1, v_h v > 0$$

$$\varphi = -\frac{\pi}{2}, \text{ for } |v_h| = v_1, v_h v < 0$$

$$r(\theta) = R_o, \text{ for } |v_h| = v_1, v_h v = 0 \quad (46c)$$

As shown in Figure 1.



**Figure 1.** Relationship on X-axis and Y-axis

### 3.3. Earth's trajectory in the solar coordinate system

According to (34), the author can know that eccentric distance  $c = \sqrt{a^2 - b^2} = 24.46 \times 10^5 km$ , eccentricity  $e = c/a = 0.01635$ , focal distance  $p = b\sqrt{1 - e^2}/e = 91.47 \times 10^8 km$

Thus, the polar equation of the earth's orbit is

$$r = \frac{ep}{1+e \cos \theta} = \frac{1.4955 \times 10^8}{1+0.01635 \cos \theta} km \quad (47)$$

### 3.4. The rocket trajectory fits the earth trajectory

Launch a rocket from the earth and make its orbit resemble the shape of the earth's orbit, phase angle difference being  $k\pi$ . To achieve the purpose of rocket return to the earth. Specifically, take the polar

coordinate system being same with formula in (45). The right end of the long axis of the earth's elliptical orbit is put into orbit, then according to the rocket, the phase and shape of the earth's orbit are consistent with conditions by comparison.

$$ep = \frac{v_h^2}{v_1^2} R_o = 1.4955 \times 10^{11} m \quad (48)$$

$$e = \frac{v_h^2}{v_1^2} \sqrt{\left(1 - \frac{v_1^2}{v_h^2}\right)^2 + \left(\frac{v}{v_h}\right)^2} = 0.01635 \quad (49)$$

$$\tan \varphi = \frac{vv_h}{v_h^2 - v_1^2} = 0 \quad (50)$$

Then, solving that

$$v_h = v_1 \sqrt{1 + e} = 29818 \sqrt{1 + 0.01635} \approx 30061 \text{ m/s} \quad (51)$$

$$v = 0 \quad (52)$$

$$R_o = \frac{ep}{1+e} = \frac{1.4955 \times 10^{11}}{1+0.01635} \approx 1.4714 \times 10^{11} \quad (53)$$

Where  $v$  and  $v_h$  are both velocities relative to the solar reference frame. Since the linear velocity of the earth in the solar system  $v_{earth} = v_1$ . Considering the reverse launch, the annular velocity of a rocket relative to the ground at launch is

$$v_{h-earth} = v_1 + v_h \approx 59879 \text{ m/s} \quad (54)$$

#### 4. Conclusion

When the earth reaches the end of its long orbit, a rocket is launched from the earth. By controlling the direction of the entry speed, the orbit of the rocket is coplanar with the earth orbit and moves in the opposite direction of the earth. In the solar reference frame, the author adjusts the orbit height of the rocket, the circumferential velocity and radial velocity are

$$R_o = 1.4714 \times 10^8 \text{ km}, v_h = 30061 \text{ m/s}, v = 0 \text{ m/s} \quad (55)$$

It will ensure that the rocket will bypass the sun and return to earth after six months. At the same time, since the rocket is required to operate in the opposite direction of the earth, the orbital velocity of the rocket relative to the sun is

$$v_{h-earth} = v_1 + v_h \approx 59879 \text{ m/s} \quad (56)$$

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