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Simulating the Influence of Time Delay on Fractional Differential Equations based on Predictor-Corrector Scheme

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Abstract. This paper aims to investigate a fractional order prey predator model with group defense and time delay through differential equation linearization theory and predictor-corrector scheme. In this model, we use the Holling-IV functional response, called Monod-Haldane function, for interactions between prey and predator species. Firstly, the uniqueness of the solution to the initial value problem of this system are proved. Secondly, the existence of equilibrium points is discussed, and the Hopf bifurcation of this system is studied using time lag as the bifurcation parameter. Finally, based on the predictor-corrector scheme, we conduct numerical simulations with corresponding parameters and different time delay parameters to analyze the impact of time delay on dynamics.

Keywords: Stability Analysis; Periodic Solution; Time Delay.

1. Introduction

Prey-predator models have been extensively studied in the fields of biology and mathematics since the seminal works of Lotka[1] and Volterra[2]. Understanding the dynamics of prey-predator interactions, including stability, persistence, periodic solutions, and bifurcation, is crucial for maintaining the balance of ecosystems. People propose different functional response functions [3, 4] to simulate animal behaviors, analyse their dynamic properties, in order to provide corresponding strategies and analysis for ecological protection, biological invasion, and other issues. In natural ecosystems, many species exhibit social behavior that can influence the behavior of other mem- bers of their population. For example, species such as wildebeest, zebras, and buffalo form herds to search for food resources and defend against predators. Based on this fact, the researchers Raw, Mishra, Kumar and Thakur [5] have proposed a prey-predator model depicted with the following differential equations:

$$\begin{cases} \dfrac{dx}{dt} = ax\left(1-\dfrac{x}{k}\right) - \dfrac{mxy}{dx^2+b}, \\[2ex] \dfrac{dy}{dt} = -py + \dfrac{\mu mxy}{dx^2+b} - \dfrac{nyz}{ey^2+c}, \\[2ex] \dfrac{dz}{dt} = -qz + \dfrac{\omega nyz}{ey^2+c}, \end{cases} \tag{1}$$

Recently researchers have been dedicating their efforts to studying the dynamic behavior of fractional-order ecosystems. These ecosystems, characterized by their fractional-order models, exhibit a unique characteristic where the next state is not only determined by the current state 1 but also influenced by all previous historical states [7]. Predators often require a certain time delay to complete digestion.

Considering the above factors, in this paper, we study the fractional order system written in the following form:

$$\begin{cases} D_*^{\alpha}x(t) = x(t)\left(a\left(1-\dfrac{x(t)}{k}\right)-\dfrac{my(t)}{dx(t)^2+b}\right) = f(x,y,z), \\ \\ D_*^{\alpha}y(t) = y(t)\left(\dfrac{\mu mx(t)}{dx(t)^2+b}-p-\dfrac{nz(t-r)}{ey(t)^2+c}\right) = g(x,y,z), \\ \\ D_*^{\alpha}z(t) = z(t-\tau)\left(\dfrac{\omega ny(t)}{ey(t)^2+c}-q\right) = h(x,y,z). \end{cases} \tag{2}$$

The paper is structured as follows: Sec. 2 investigates the existence and uniqueness of the system in a fractional order framework. Sec. 3 analyzes the equilibrium point of the system (1.2). Sec. 4 explores the influence of time delay through numerical simulations. Finally, Sec. 5 presents a brief conclusion.

2. Existence and Uniqueness of Solution

Definition 1. *[6] The Caputo fractional derivative with order* $q \geq 0$ *for the continuous function* $g(t) \in \mathcal{A}C^n([0,+\infty),\mathbb{R})$ is defined as

$$ {}_0^C D_t^q g(t) = \frac{1}{\Gamma(n-q)}\int_0^t \frac{g^{(n)}(s)}{(t-s)^{q-n+1}}ds, \tag{3}$$

where $q \in (n-1, n)$ *and* $n \in \mathbb{Z}_+$, $\Gamma(\cdot)$ *is the Gamma function.*

In particular, when $n = 1$, i.e., $0 < q \leq 1$, the Caputo fractional derivative with order q becomes

$$ {}_0^C D_t^q g(t) = \frac{1}{\Gamma(1-q)}\int_0^t \frac{g(s)}{(t-s)^{q}}ds.$$

Lemma 1. [9] For system

$$ {}_0^C D_t^q X(t) = \Psi(t,X), t \geq 0$$

with initial condition $X(0) = (x(0), y(0))$*, where* $0 < q \leq 1$, $\Psi : [0,\infty) \times \Delta \to R_n, \Delta \subseteq \mathbb{R}_n$, *if* $\Psi(t, X)$ *fulfills the local Lipschitz condition about* $X \in \mathbb{R}_n$*:*

$$\| \Psi(t,X) - \Psi(t,\tilde{X}) \| \leq M \| X - \tilde{X} \|$$

then the system exists a unique solution on $[0, \infty) \times \Delta$, and $\| X(x_1,x_2,\cdots\cdots x_n) - \tilde{X}(\tilde{x}_1,\tilde{x}_2,\cdots\cdots \tilde{x}_n) \| \leq \sum_{i=1}^n |x_i - \tilde{x}_i|, i = 1,2,\cdots\cdots n, x_i, \tilde{x}_i \in R$

Lemma 2. *[11]* Assume $q > 0$, $\beta > 0$ and $K \in C^{n\times n}$,then

$$\mathcal{L}\left\{t^{\beta-1}E_{q,\beta}\left(Kt^q\right)\right\} = \frac{s^{q-\beta}}{s^q - K}$$

for $\mathrm{Re}(s) > \|K\|^{\frac{1}{q}}$, where $\mathrm{Re}(s)$ is the real part of the complex number s and Eq, β is the Mittag-Leffler function.

Lemma 3. *[12]* For the following fractional-order system:

$$ {}_0^C D_t^q g(t) = h(g(t)), \quad g(0) = g_0 \in \mathbb{R}^N, \quad q \in (0,1), $$

where $g(t) = \left(g_1(t), g_2(t), \ldots, g_n(t)\right)^T \in \mathbb{R}^n$ *and* $h: \left[h_1, h_2, \ldots, h_n\right]: \mathbb{R}^n \to \mathbb{R}^n$ *If* $h\left(g^*\right) = 0$, *then* $g*$ is an equilibrium point. Set J ($g*$) to be the Jacobian matrix $J = \frac{\partial h}{\partial g} = \frac{\partial\left(h_1, h_2, \ldots, h_n\right)}{\partial\left(g_1, g_2, \ldots, g_n\right)}$ $\partial(g1\ ,g2\ ,...,gn)$ for $g = g*$. If the characteristic values λi (i = 1, ... , n) of J ($g*$) meet $|\arg(\lambda i\ )| > q\pi\ 2$ (i = 1, … , n), then $g\ *$ is locally asymptotically stable.

Theorem 1. For each $X_{t0} = (x_{t0}, y_{t0}, z_{t0}) \in \Omega$, there exists a unique solution $X(t) \in \Omega$ of system (1.2) with initial condition X_{t0} , which is defined for all $t \geq t0$.

Proof. We study the existence and uniqueness of the solution of system (1.2) in the region $\Omega \times [t0\ , T\ ]$, where $\Omega = \{(x, y, z) \in \mathrm{R}^3\colon \max\{|x|, |y|, |z|\} < M\}$ and $T < +\infty$. Define a mapping $H(X) = (H1\ (X), H2\ (X), H3\ (X))$, in which

$$
\begin{aligned}
H_1(X) &= \left| x(t)\left(a\left(1 - \frac{x(t)}{k}\right) - \frac{my(t)}{dx(t)^2 + b} \right) \right| \\
H_2(X) &= \left| y(t)\left(\frac{\mu mx(t)}{dx(t)^2 + b} - p - \frac{nz(t-\tau)}{ey(t)^2 + c} \right) \right| \\
H_3(X) &= \left| z(t-\tau)\left(\frac{\omega ny(t)}{ey(t)^2 + c} - q \right) \right|
\end{aligned}
$$

For any X, where $\overline{X} \in \Omega$, it follows from above that

$$
\begin{aligned}
&\| H(X) - H(\overline{X}) \| = \left|H_1(X) - H_1(\overline{X})\right| + \left|H_2(X) - H_2(\overline{X})\right| + \left|H_3(X) - H_3(\overline{X})\right| \\
&= \left| x(t_1)(a(1 - \frac{x(t_1)}{k}) - \frac{my(t_1)}{dx_1^2 + b}) - x(t_2)(a(1 - \frac{x(t_2)}{k}) - \frac{my(t_2)}{dx_2^2 + b}) \right| \\
&+ \left| y(t_1)(\frac{\mu mx(t_1)}{dx(t_1)^2 + b} - p - \frac{nz(t_1 - \tau)}{ey(t_1)^2 +_c}) - y(t_2)(\frac{\mu mx(t_2)}{dx(t_2)^2 + b} - p - \frac{nz(t_2 - \tau)}{ey(t_2)^2 +_c}) \right| \\
&+ \left| z(t_1 - \tau)(\frac{wny(t_1)}{ey(t_1)^2 + c} - q) - z(t_2 - \tau)(\frac{wny(t_2)}{ey(t_2)^2 + c} - q) \right| \\
&\leq | x(t_1) - x(t_2) | \left| a\left(1 - \frac{x(t_1)}{k}\right) - \frac{my(t_1)}{dx^2(t_1) + b} \right| \\
&+ \max\left(\left|x(t_1)\right|, \left|x(t_2)\right|\right) \left| a\left(1 - \frac{x(t_1)}{k}\right) - \frac{my(t_1)}{dx^2(t_1) + b} - a\left(1 - \frac{x(t_2)}{k}\right) + \frac{my(t_2)}{dx^2(t_2) + b} \right| \\
&+ \left|y(t_1) - y(t_2)\right| \left| \frac{\mu mx(k_1)}{dx(t_1) + b} - p - \frac{nz(t_1 - \tau)}{ey(t_2)^2 + c} \right| + \max\left(\left|y(t_1)\right|, \left|y(t_2)\right|\right) \times \\
&\left| \frac{\mu mx(t_1)}{dx(t_2)^2 + b} - \frac{\mu mx(t_2)}{dx(t_2)^2 + b} + \frac{nz(t_1 - \tau)}{c} + \frac{nz(t_2 - \tau)}{c} \right| + \left|z(t_1 - \tau) - z(t_2 - \tau)\right| \left| \frac{why(t_1)}{ey(c_1) + c} - q \right| \\
&+ \max\left(| z(t - \tau) |, \left|z(t_0 - \tau)\right|\right) \left| \frac{wny(t_0)}{ey(t_1)^2 + c} - \frac{wny(t_0)}{ey(t_2)^2 + c} \right| \\
&\leq \left|x(t_1) - x(t_2)\right| \left| a\left(1 - \frac{x(t_1)}{k}\right) - \frac{my(t_1)}{dx^2(t_1) + b} \right| + M \max\left(\left|x(t_1) - x(t_2)\right|, \left|y(t_1) - y(t_2)\right|, | z(t_1) - z(t_2) |\right) \times \\
&\left| a\left(1 - \frac{x_1}{k}\right) + \frac{my_1}{b} \right| + \left|y(t_1) - y(t_2)\right| \left| \frac{\mu mx(t_1)}{dx\left(t_1^2\right)^2 + b} - p - \frac{nz(t_1 - \tau)}{ey(t_1)^2 + c} \right| + M \max\left(\left|x(t_1) - x(t_2)\right|, | y(t_1)\right. \\
&\left. - y(t_2) |, | z(t_1 - \tau)\right| - z(t_2 - \tau)\left|\right) \left| \frac{\mu mx_2}{dx_2 + b} - \frac{nz_2}{c} \right| + \left|z(t_1 - \tau) - z(t_2 - \tau)\right| \left| \frac{wy(t_1)}{ey(t_1)^2 + c} - q \right| \\
&M \max\left(\left|x(t_1) - x(t_2)\right|, \left|y(t_1) - y(t_2)\right|, \left|z(t_1 - \tau) - z(t_2 - \tau)\right|\right) \left| \frac{wny_3}{ey_3^2 + c} \right| \\
&\leq L\| X - \overline{X}\| ,
\end{aligned}
$$

where

$$L = \max\left\{\left|a\left(1-\frac{(t_1)}{k}\right)-\frac{my(t_1)}{dx^2(t_1)+b}\right|+M\left(\left|a\left(1-\frac{x_1}{k}\right)-\frac{my_1}{b}\right|+\left|\frac{\mu mx_2}{dx_2+b}-\frac{nz_2}{c}\right|+\left|\frac{wny_3}{eg_3^2+c}\right|\right)\right.$$
$$\left|\frac{\mu mx(t_1)}{dx(t_1)^2+b}-p-\frac{nz(t_1-\tau)}{\rho g(t_1)^2+c}\right|+M\left(\left|a\left(1-\frac{x_1}{k}\right)-\frac{my_1}{b}\right|+\left|\frac{\mu mx_2}{dx_2+b}-\frac{nz_2}{c}\right|+\left|\frac{wny_3}{eg_3^2+c}\right|\right),$$
$$\left.\left|\frac{wny(t_1)}{ey(t_1)+c}-q\right|++M\left(\left|a\left(1-\frac{x_1}{k}\right)-\frac{my_1}{b}\right|+\left|\frac{\mu mx_2}{dx_2+b}-\frac{nz_2}{c}\right|+\left|\frac{wny_3}{eg_3^2+c}\right|\right)\right\}.$$

Thus, $H(X)$ satisfies the Lipschitz condition with respect to X. This indicates there exists a unique solution $X(t)$ of system (1.2) with initial condition $X_{t0} = (x_{t0}, y_{t0}, z_{t0}) \in \Omega$

3. Equilibrium Point Analysis

In this section, we investigate the existence and stability of the equilibrium points of system (1.2). Equilibrium points of system (1.2) are given as follows:

(i) The trivial equilibrium point $E0 = (0, 0, 0)$ always exists.

(ii) The prey-only-equilibrium point $E1 = (k, 0, 0)$ always exists.

(iii) In the absence of a top predator, the boundary equilibria are given by $\bar{E} = (\bar{x}, \bar{y}, 0)$ in the interior of positive quadrant of $x - y$ plane, where

$$\bar{x} = \frac{I \pm \sqrt{I^2 - 4P}}{2}$$
$$\bar{y} = \frac{a}{m}\left(1-\frac{\bar{x}}{k}\right)\left(d\bar{x}^2+b\right)$$

These equilibria exist if $I^2 > 4P$ and $k > \bar{x}$.

(iv) The interior equilibria $\hat{E} = (\hat{x}, \hat{y}, \hat{z})$ exist if and only if $(\hat{x}, \hat{y}, \hat{z})$ is a positive solution of the following system

$$\begin{cases} g_1 = a\left(1-\dfrac{x}{k}\right)-\dfrac{my}{dx^2+b} = 0 \\ g_2 = -p+\dfrac{\mu mx}{dx^2+b}-\dfrac{nz}{ey^2+c} = 0 \\ g_3 = -q+\dfrac{\omega ny}{ey^2+c} = 0 \end{cases}$$

From $g3 = 0$, we obtain

$$\hat{y} = \frac{Q \pm \sqrt{Q^2 - 4S}}{2}.$$

In addition, from the first equation of system (3), we have the following cubic equation:

$$f(x) = Ax^3 + Bx^2 + Cx + E = 0$$

in which

$$A = 1, B = -k, C = \frac{b}{a}, E = \frac{k}{d}(\frac{m}{a}\hat{y} - b)$$

Now we show that Eq. (3) has a positive root. If $\hat{y} < \frac{ab}{m}$, then we conclude that

$$f(0) = \frac{k}{d}(\frac{m}{a}\hat{y} - b) < 0,$$
$$f(k) = \frac{k}{d}(\frac{m}{a}\hat{y}) < 0,$$

which then further implies that $f(0)\, f(k) < 0$. Clearly Eq. (3) has a positive root $\hat{x} \in (0, k)$ when $\hat{y} < \frac{ab}{m}$. Also, we deduce that

$$\hat{z} = \frac{1}{n}\left(-p + \frac{\mu m \hat{x}}{d\hat{x}^2 + b}\right)\left(e\hat{y}^2 + c\right)$$

Therefore, the interior equilibria E^* exist under the following conditions

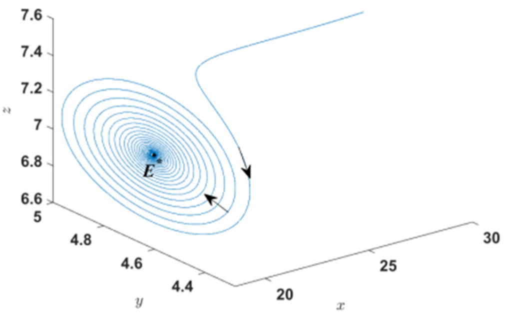
$$Q^2 > 4S, \hat{y} < \frac{ab}{m},$$
$$d\hat{x} + \frac{b}{\hat{x}} < \frac{m\mu}{p}.$$

4. Numerical Simulation and Analysis

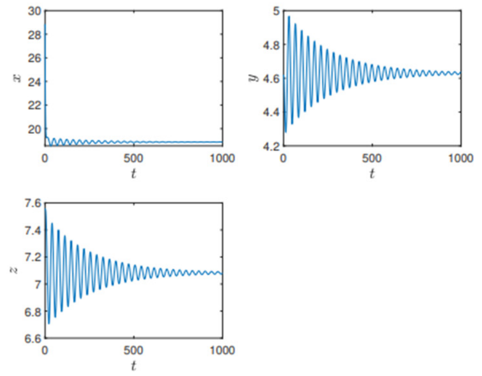
In this section, we will provide some numerical simulations through a predictor-corrector scheme [8]. We studied the dynamical behavior of system (1.2) for different parameter values. We fix the values of the parameters except for k and the order a at

$$\mathrm{a} = 0.7, \mathrm{m} = 0.65, \mathrm{d} = 0.025,$$
$$\mathrm{b} = 15, \mathrm{m} = 0.9, \mathrm{p} = 0.23, \mathrm{n} = 0.45, \mathrm{e} = 0.035,$$
$$\mathrm{c} = 13, \mathrm{w} = 0.99, \mathrm{q} = 0.15.$$

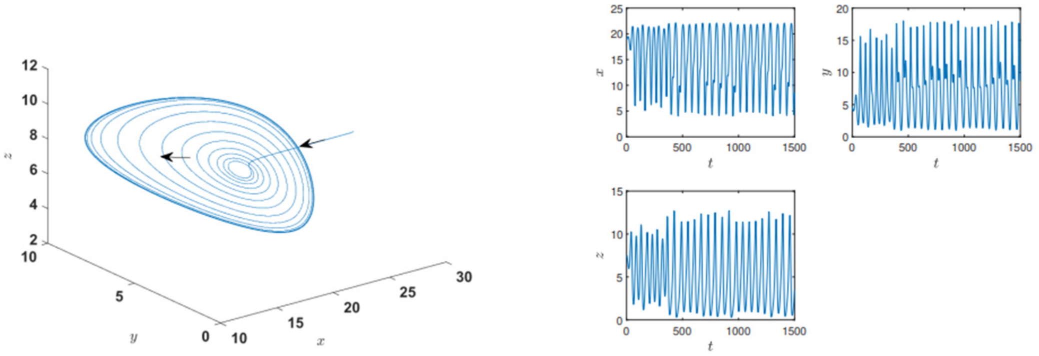
We then change the parameter k



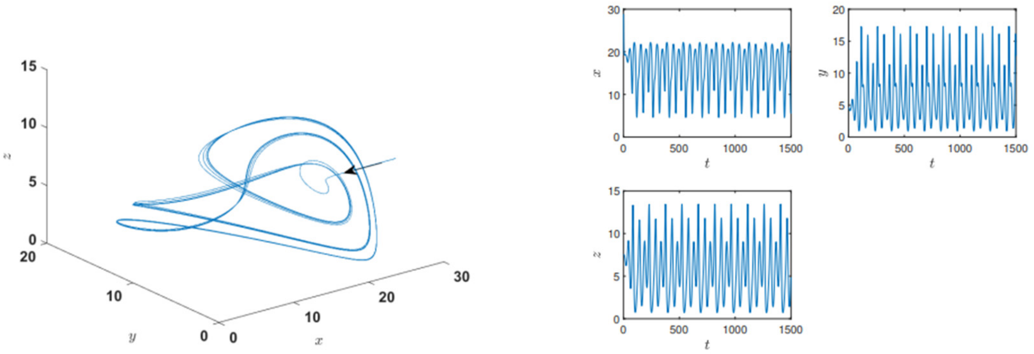
(a) $\tau = 0$



(b) $\tau = 0$



(c) τ =0.8 (d) τ = 0.8



(e) τ = 3.5 (f) τ = 3.5

Figure 1. The figure on the left is the phase diagram for a = 0.98, k = 23. The figure on the right is the time series diagram corresponding to it

Fig. 1 shows the phase diagram and time series plot for system (1.2) at a = 0.98, k = 23. When τ = 0, the equilibrium point is stable, and orbits from any initial point will be attracted to E∗ , as shown in subFig.s (a,b). When τ = 0.8, the equilibrium point is destabilized, and a stable limit cycle appears around it. At this point, point trajectories from the ring are attracted to the permutation, as shown in subFig.s (c,d). As T continues to increase to 3.5, the shape of the ring attractor changes, taking on a shape that appears to cross itself but does not actually do so, as shown in subFig. (e, f).

5. Conclusion

This paper studies the dynamic behavior of three animals' fractional order predation system and their group defense mechanism system. Considering the time delay effect of the third population, a new system is established by introducing a time delay term. The existence of the equilibrium point of the system is analyzed. Finally, numerical simulations are carried out for specific param-eter values, with system phase and time series diagrams. It is found that the initially unstable phenomenon becomes stable after introducing the time lag term. It is possible to adjust the size of the time lag to obtain the conditions for system stabilization, cyclic phenomena, and the generation of chaotic phenomena. Therefore, the time lag is essential in controlling chaos in dynamic systems.

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