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High-order total differential solution of multivariate function based on Yang Hui triangle

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Abstract. Under the current study, there are fewer ways to solve the higher-order full differential of multivariate functions. This paper introduces Yanghui Triangle and the higher-dimensional form of Yanghui Triangle to study the field of higher-order full differential solution of multivariate functions, to summarize the general formulas and to improve the efficiency of operation. Based on the assumptions of infinite partial derivatives and continuity of partial derivatives, this paper first proves the correctness of the higher-order total differentiation formula for binary functions by mathematical induction and tests it with examples, and the result proves that the formula is valid and has good applicability. Then, the higher-order total differential formula for ternary functions and more multivariate functions are mathematically inducted and proved to obtain a generalized formula, which improves the computational efficiency. The generalized formulas studied in this paper assume that multivariate functions can have partial derivatives and the partial derivatives are continuous, which is consistent with the assumptions of full differential computation of multivariate functions in practical applications of science and engineering technology. However, for the case where the partial derivatives of multivariate functions are not continuous, it needs to be proved in future studies.

Keywords: Pascal's triangle; mathematical induction; chain rule for derivatives; high-order full differentials.

1. Introduction

For the high-order total differential problem of multivariate functions, conventional method is to use chain rules for derivation. However, when such methods are faced with the calculation of high-order total differential of multivariate functions, if the multivariate function has no special properties, such as rotation symmetry, low-order power functions, etc., a large amount of calculation is required, and the chain rule derivation is performed. The computational efficiency is low.

Yang Hui triangle, also known as Pascal triangle, was invented by Yang Hui, a mathematician in the Southern Song Dynasty of China. The basic principle of Yang Hui triangle is to generate its internal numbers through recursive relations and combinatorial number properties. Its basic principle is consistent with the binomial theorem created by Newton later. Based on many attempts to obtain the higher order differential of the binary function, this paper finds a general rule, that is, the coefficients of the Yang Hui triangle are consistent with the coefficients of the chain rule derivation. Subsequently, the formula is extended to a higher dimension, that is, by using the high-dimensional Yang Hui triangle, also known as Newton polynomial expansion, the general formula for calculating the high-order total differential of n-ary n-dimensional functions is obtained.

In this paper, the high-dimensional form of Yang Hui triangle and Yang Hui triangle is used to calculate the high-order total differential of multivariate function, and the mathematical induction method is used for induction and deduction, also, the universal formula is obtained.

2. Concept review

2.1. Yang Hui Triangle Introduction

Yang Hui triangle [1-2] is a concept put forward by Yang Hui, a mathematician in the Southern Song Dynasty of China. The mathematical principle contained in Yang Hui triangle is like Newton 's binomial theorem [3], which is widely used in the field of higher-order equation expansion.

The main properties of Yang Hui triangle are as follows:

First: Each number in the Yang Hui triangle is equal to the sum of two numbers above it.

$$C_{n+1}^{m} = C_n^m + C_n^{m-1} (n, m \in N^*, m \geq 1) \tag{1}$$

Second: The sum of all the numbers in row n of Yang Hui triangle is 2^n.

$$C_n^0 + C_n^1 + C_n^2 + \cdots C_n^n = 2^n \tag{2}$$

Third: The number of M and the number of $N - M + 1$ in the N row of Yang Hui triangle is equal.

$$C_n^m = C_n^{n-m} \tag{3}$$

The above introduction is the two-dimensional form of Yang Hui triangle. The following introduces the high-dimensional form of Yang Hui triangle, it also referred as Newton polynomial expansion. The mathematical principle is as follows:

$$(a_1 + a_2 + \cdots a_n)^n = \binom{n}{k_1, k_2, \ldots k_n} a_1^{k_1} a_2^{k_2} \ldots a_n^{k_n} \tag{4}$$

Among them, $k_1, k_2, \ldots k_n = n, (k_i, i \in N^*)$, $\binom{n}{k_1, k_2, \ldots k_n}$ represents the number of combinations of n tuple $(k_i, i \in N^*)$.

2.2. Introduction of total differential of multivariate function

Let [4] $z = f(x, y)$ have a continuous partial derivative on the domain $D \subset R^2$, which is differentiable and satisfies

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \tag{5}$$

Such a class of equations is called the first-order total differential of z. If z has a high-order continuous partial derivative, and $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are differentiable in high order, it can be extended to $d^k z$ differentiable, and the differential is called high-order differential.

The traditional method for solving the total differential of multivariate function is in the form of a chain rule. The mathematical formula is shown in formula (6) and formula (7) [5].

$$d^k z = (dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y})^k z(k = 1,2\cdots) \tag{6}$$

$$d^k u = (dx_1\frac{\partial}{\partial x_1} + dx_2\frac{\partial}{\partial x_2} + \cdots + dx_n\frac{\partial}{\partial x_n})^k u, (k = 1,2\cdots) \tag{7}$$

However, when this kind of calculation method is used to calculate the high-order total differential formula of binary function, if the calculation function has no special properties, the calculation process is complicated. For example, for the fourth-order total differential of multivariate function $u = xe^y$, it is necessary to first derive the chain rule, then derive the partial derivative of the function, and finally obtain the result. The calculation is complex and inefficient. Now, the general solution process of Yang Hui triangle and high-order total differential of multivariate function is adopted, and the calculation process is simplified by using the formula obtained by mathematical induction [6].

3. Proof of the formula of higher order total differential of binary function

3.1. Higher order total differential formula of binary function

Let $u = u(x, y)$ be infinitely partial derivative and partial derivative continuous

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{8}$$

$$d^2 u = \frac{\partial^2 u}{\partial x^2}dx^2 + \frac{\partial^2 u}{\partial x \partial y}dxdy + \frac{\partial^2 u}{\partial y^2}dy^2 + \frac{\partial^2 u}{\partial y \partial x}dydx \tag{9}$$

By the third edition of mathematical analysis (Chen, J.X, Yu, C.H, Jin. L) P130 [7]: In the practical application of science and engineering technology, it is often considered that the partial derivatives are continuous, so do not mind the order of partial derivatives, thus the (9) is sorted out.

The formula is shown in (10):

$$d^2 u = \frac{\partial^2 u}{\partial x^2}dx^2 + 2\frac{\partial^2 u}{\partial x \partial y}dxdy + \frac{\partial^2 u}{\partial y^2}dy^2 \tag{10}$$

Find the third-order total differential u and arrange it as shown in (12):

$$d^3 u = \frac{\partial^3 u}{\partial x 3}dx^3 + \frac{\partial^3 u}{\partial x^2 \partial y}dx^2 dy + \frac{\partial^3 u}{\partial y^3}dy^3 + \frac{\partial^3 u}{\partial y^2 \partial x}dy^2 dx + 2\frac{\partial^3 u}{\partial x^2 \partial y}dx^2 dy + 2\frac{\partial^3 u}{\partial x \partial y^2}dxdy^2 \tag{11}$$

$$d^3 u = \frac{\partial^3 u}{\partial x^3}dx^3 + 3\frac{\partial^3 u}{\partial x^2 \partial y}dx^2 dy + \frac{\partial^3 u}{\partial y^3}dy^3 + 3\frac{\partial^3 u}{\partial y^2 \partial x}dy^2 dx \tag{12}$$

Similarly, the fourth-order total differential of u is solved and sorted out as shown in (13):

$$d^4 u = \frac{\partial^4 u}{\partial x^4}dx^4 + 4\frac{\partial^4 u}{\partial x^3 \partial y}dx^3 dy + \frac{\partial^4 u}{\partial y^4}dy^4 + 4\frac{\partial^4 u}{\partial y^3 \partial x}dy^3 dx + 6\frac{\partial^4 u}{\partial x^2 \partial y^2}dx^2 dy^2 \tag{13}$$

By observing the formulas (9) to (13), the general formula for solving the high-order total differential of the binary function can be obtained by mathematical induction. The reasoning result is as shown in formula (14):

$$d^n u = C_n^0 \frac{\partial^n u}{\partial x^n} dx^n + C_n^1 \frac{\partial^n u}{\partial x^{n-1} \partial y} dx^{n-1} dy + C_n^2 \frac{\partial^n u}{\partial x^{n-2} \partial y^2} dx^{n-2} dy^2 + \cdots\cdots + C_n^n \frac{\partial^n u}{\partial y^n} dy^n \quad (14)$$

Where: $C_n^i (0 \le i \le n)$ is the binomial coefficient.

To use mathematical induction from (9) to (14), it is necessary to prove that (14) is also true for $d^{n+1}u$:
From the properties of Yang Hui triangle:

$$C_{n+1}^i = C_n^i + C_n^{i-1} (1 \le i \le n) \quad (15)$$

By analyzing Equation (14), it can be seen from the chain derivative rule [8] that for any order partial derivative $\frac{\partial^m u}{\partial x^p \partial y^{m-p}}$, there are only two results of $\frac{\partial^{m+1} u}{\partial x^{p+1} \partial y^{m-p-1}}$ and $\frac{\partial^{m+1} u}{\partial x^p \partial y^{m-p+1}}$. Since $u = u(x, y)$ is infinitely partial derivative and partial derivative is continuous, $\frac{\partial^{m+1} u}{\partial y^{p+1} \partial x^{m-p-1}}$, $\frac{\partial^{m+1} u}{\partial y^p \partial x^{m-p+1}}$ and $\frac{\partial^{m+1} u}{\partial x^{p+1} \partial y^{m-p-1}}$, $\frac{\partial^{m+1} u}{\partial x^p \partial y^{m-p+1}}$ are equivalent, and their coefficient sum satisfies Formula (1). It is easy to prove that $\frac{\partial^n u}{\partial x^n}$ and $\frac{\partial^n u}{\partial y^n}$ have only one path (that is, continuous n -order partial derivative x, y), and their coefficients are always equal to 1. The following is based on the above analysis to prove:

Firstly, it is proved that $\frac{\partial u^n}{\partial x^n}$ and $\frac{\partial u^n}{\partial y^n}$ have only one path:

Assume that $\frac{\partial^n u}{\partial x^n}$ and $\frac{\partial u^n}{\partial y^n}$ are not only obtained by a unique path, then there must be a partial derivative of u two y or x, and there must be such items as $\frac{\partial^n u}{\partial x^{n-1} \partial y}$ -shaped, so the hypothesis is not valid.

Proof:

$\frac{\partial^n u}{\partial x^n}$ and $\frac{\partial^n u}{\partial y^n}$ have only one path.

$$\begin{aligned} d^{n+1} u = {} & C_{n+1}^0 \frac{\partial^{n+1} u}{\partial x^{n+1}} dx^{n+1} + (C_n^0 + C_n^1) \frac{\partial^{n+1} u}{\partial x^n \partial y} dx^n dy + \\ & (C_n^2 + C_n^1) \frac{\partial^{n+1} u}{\partial x^{n-1} \partial y^2} dx^{n-1} dy^2 + \cdots\cdots + C_{n+1}^{n+1} \frac{\partial^{n+1} u}{\partial y^{n+1}} dy^{n+1} \end{aligned} \quad (16)$$

$$d^{n+1}u = C_{n+1}^{0}\frac{\partial^{n+1}u}{\partial x^{n+1}}dx^{n+1} + (C_{n+1}^{1})\frac{\partial^{n+1}u}{\partial x^{n}\partial y}dx^{n}dy + (C_{n+1}^{2})\frac{\partial^{n+1}u}{\partial x^{n-1}\partial y^{2}}dx^{n-1}dy^{2} + \cdots\cdots + C_{n+1}^{n+1}\frac{\partial^{n+1}u}{\partial y^{n+1}}dy^{n+1} \tag{17}$$

The formula of Formula (17) is decomposed by chain rule path, and the process of each term in the formula is reproduced. By using Yang Hui 's triangle property (as shown in Formula (9)), the formula (7) can be obtained by decomposing item by item. Finally, the formula is proved.

$$d^{n}u = C_{n}^{0}\frac{\partial^{n}u}{\partial x^{n}}dx^{n} + C_{n}^{1}\frac{\partial^{n}u}{\partial x^{n-1}\partial y}dx^{n-1}dy + C_{n}^{2}\frac{\partial^{n}u}{\partial x^{n-2}\partial y^{2}}dx^{n-2}dy^{2} + \cdots\cdots + C_{n}^{n}\frac{\partial^{n}u}{\partial y^{n}}dy^{n} \tag{18}$$

3.2. Application of higher order total differential formula of binary function

The previous part proves the high-order total differential formula of the binary function. Now, three examples are applied. The derivative functions in the problem satisfy $u = u(x, y)$ infinite partial derivatives and partial derivatives are continuous.

Example 1: $u = e^{x} \times e^{y}$, $u = u(x, y)$ are infinitely partial derivative and partial derivative is continuous. Find $d^{4}u$.

To test the reliability of the formula, we first use the conventional method to solve, and then use the formula to verify.

The solution process of the conventional method is as follows:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{19}$$

$$d^{2}u = \frac{\partial^{2}u}{\partial x^{2}}dx^{2} + 2\frac{\partial^{2}u}{\partial x\partial y}dxdy + \frac{\partial^{2}u}{\partial y^{2}}dy^{2} \tag{20}$$

$$d^{3}u = \frac{\partial^{3}u}{\partial x^{3}}dx^{3} + 3\frac{\partial^{3}u}{\partial x^{2}\partial y}dx^{2}dy + \frac{\partial^{3}u}{\partial y^{3}}dy^{3} + 3\frac{\partial^{3}u}{\partial y^{2}\partial x}dy^{2}dx \tag{21}$$

$$d^{4}u = \frac{\partial^{4}u}{\partial x^{4}}dx^{4} + 4\frac{\partial^{4}u}{\partial x^{3}\partial y}dx^{3}dy + \frac{\partial^{4}u}{\partial y^{4}}dy^{4} + 4\frac{\partial^{4}u}{\partial y^{3}\partial x}dy^{3}dx + 6\frac{\partial^{4}u}{\partial x^{2}\partial y^{2}}dx^{2}dy^{2} \tag{22}$$

Solution:

$$I_{1} = d^{4}u = e^{x+y}dx^{4} + 4e^{x+y}dx^{3}dy + e^{x+y}dy^{4} + 4e^{x+y}dy^{3}dx + 6e^{x+y}dx^{2}dy^{2} \tag{23}$$

Using the formula (18) to solve:

$$I_{2} = d^{4}u = e^{x+y}dx^{4} + 4e^{x+y}dx^{3}dy + 6e^{x+y}dx^{2}dy^{2} + 4e^{x+y}dy^{3}dx + e^{x+y}dy^{4} \tag{24}$$

Arranged, apparently

$$I_1 = I_2 \tag{25}$$

Example 2: $u = x^5 e^y$, $u = u(x, y)$ are infinitely partial derivative and partial derivative is continuous. Find d^4u

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{26}$$

$$d^2u = \frac{\partial^2 u}{\partial x^2}dx^2 + 2\frac{\partial^2 u}{\partial x \partial y}dxdy + \frac{\partial^2 u}{\partial y^2}dy^2 \tag{27}$$

$$d^3u = \frac{\partial^3 u}{\partial x^3}dx^3 + 3\frac{\partial^3 u}{\partial x^2 \partial y}dx^2dy + \frac{\partial^3 u}{\partial y^3}dy^3 + 3\frac{\partial^3 u}{\partial y^2 \partial x}dy^2dx \tag{28}$$

$$d^4u = \frac{\partial^4 u}{\partial x^4}dx^4 + 4\frac{\partial^4 u}{\partial x^3 \partial y}dx^3dy + \frac{\partial^4 u}{\partial y^4}dy^4 + 4\frac{\partial^4 u}{\partial y^3 \partial x}dy^3dx + 6\frac{\partial^4 u}{\partial x^2 \partial y^2}dx^2dy^2 \tag{29}$$

Solution:

$$I_1 = d^4u = 120xe^y dx^4 + 240x^2e^y dx^3dy + x^5e^y dy^4 + 20x^4e^y dy^3dx + 120x^3e^y dx^2dy^2 \tag{30}$$

Using the formula (18) to solve:

$$I_2 = d^4u = 120xe^y dx^4 + 240x^2e^y dx^3dy + 120x^3e^y dx^2dy^2 + 20x^4e^y dy^3dx + x^5e^y dy^4 \tag{31}$$

Arranged, apparently

$$I_1 = I_2 \tag{32}$$

Example 3: $u = xe^y$, $u = u(x, y)$ infinitely partial derivative and partial derivative continuous, find d^4u

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{33}$$

$$d^2u = \frac{\partial^2 u}{\partial x^2}dx^2 + 2\frac{\partial^2 u}{\partial x \partial y}dxdy + \frac{\partial^2 u}{\partial y^2}dy^2 \tag{34}$$

$$d^3u = \frac{\partial^3 u}{\partial x^3}dx^3 + 3\frac{\partial^3 u}{\partial x^2 \partial y}dx^2dy + \frac{\partial^3 u}{\partial y^3}dy^3 + 3\frac{\partial^3 u}{\partial y^2 \partial x}dy^2dx \tag{35}$$

Solution:

$$d^4u=\frac{\partial^4 u}{\partial x^4}dx^4+4\frac{\partial^4 u}{\partial x^3\partial y}dx^3dy+\frac{\partial^4 u}{\partial y^4}dy^4+4\frac{\partial^4 u}{\partial y^3\partial x}dy^3dx+6\frac{\partial^4 u}{\partial x^2\partial y^2}dx^2dy^2 \quad (36)$$

Using the formula (18) to solve:

$$d^4u=4e^ydy^3dx+xe^ydy^4 \quad (37)$$

Arranged, apparently:

$$I_1=I_2 \quad (38)$$

According to example one, example two, and example three, it is easy to find that using the high-order total differential formula of binary function, some fragments in high-order differential can be quickly extracted, avoiding lengthy mathematical induction derivation and chain rule derivation. For special properties, such as a partial derivative 0 after experiencing high-order differential, such properties are common in power functions, which can be used to quickly solve high-order total differential. Compared with the original chain rule derivation calculation method, this kind of calculation method can effectively improve the calculation efficiency by applying the formula.

4. Higher order total differential formula from ternary function to multivariate function

Suppose $u=u(x,y,z,z_1,z_2,\dots z_n\dots)$, the function is derivable infinitely many times and the partial derivatives are continuous. Mathematical induction is used for the high-order total differential formula of the ternary function and the high-order total differential formula of the multivariate function. The derivation process is as follows:

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz \quad (39)$$

$$\begin{aligned}d^2u&=\frac{\partial^2 u}{\partial x^2}dx^2+\frac{\partial^2 u}{\partial x\partial y}dxdy+\frac{\partial^2 u}{\partial x\partial z}dxdz+\frac{\partial u^2}{\partial y^2}dy^2+\\&\frac{\partial^2 u}{\partial y\partial x}dydx+\frac{\partial^2 u}{\partial y\partial z}dydz+\frac{\partial^2 u}{\partial z^2}dz^2+\frac{\partial u^2}{\partial x\partial z}dxdz+\frac{\partial u^2}{\partial y\partial z}dydz\\&=\frac{\partial^2 u}{\partial x^2}dx^2+2\frac{\partial^2 u}{\partial x\partial y}dxdy+2\frac{\partial^2 u}{\partial x\partial z}+2\frac{\partial^2 u}{\partial y\partial z}dydz+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\end{aligned} \quad (40)$$

The coefficients of observation (39) were 1, 1, 1 ; (40)equation coefficient is 1, 2, 2, 2, 1, 1, (41) equation coefficient is 1, 3, 3, 3, 6, 3, 3, 3, 1, 1, (42)equation coefficient is 1, 4, 4, 4, 6, 6, 6, 12, 12, 12, 4, 4, 4, 4, 1, 1; by observing the above three-formula coefficients, it is found that the structure is similar to that of the three-dimensional Yang Hui triangular array[9-10]. Therefore, mathematical induction is used for both the dimension and the number of total differentials:

$$d^nu=\sum\binom{n}{k_1,k_2,\dots,k_n}\frac{\partial^n u}{\partial x^{k_r}\partial y^{k_j}\partial z^{k_k}} \quad (41)$$

Among them, $k_i, k_j, k_k \in N*, i, j, k \in N, k_i + k_j + k_k = n$, $\begin{pmatrix} n \\ k_1, k_2, \ldots k_n \end{pmatrix}$ represents the number of combinations of n tuples $(k_i, i \in N^*)$.

Proof $d^{n+1}u$ process isomorphism formula (18).

5. Conclusion

From the above examples, the application of formula $d^n u = C_n^0 \frac{\partial^n u}{\partial x^n} dx^n + C_n^1 \frac{\partial^n u}{\partial x^{n-1}\partial y} dx^{n-1}dy + C_n^2 \frac{\partial^n u}{\partial x^{n-2}\partial y^2} dx^{n-2}dy^2 + \cdots\cdots + C_n^n \frac{\partial^n u}{\partial y^n} dy^n$ reduces the calculation amount of the conventional chain derivation rule when solving high-order differential equations, integrates the lengthy derivation term with continuous partial derivatives, the order of partial derivatives and the conclusion that the results have no effect, and reduces the complexity of the arithmetic operation.

The advantage of this formula is that for any k --order partial derivative and partial derivative continuous function, if there is an item with the partial derivative of 0 after order derivation, the operation can be further simplified and the meaningful item can be solved quickly. If x, y are assigned, the value of k -order total differential is calculated. By using formula $d^n u = C_n^0 \frac{\partial^n u}{\partial x^n} dx^n + C_n^1 \frac{\partial^n u}{\partial x^{n-1}\partial y} dx^{n-1}dy + C_n^2 \frac{\partial^n u}{\partial x^{n-2}\partial y^2} dx^{n-2}dy^2 + \cdots\cdots + C_n^n \frac{\partial^n u}{\partial y^n} dy^n$, the lengthy high-order total differential formula can be simplified into a form that can directly participate in the operation, and the operation efficiency can be improved.

This formula combines the low-dimensional and high-dimensional Yang Hui triangle with the chain rule, analyzes the general structure of the partial differential part of the chain rule, then uses the mathematical induction method to solve the general formula of the high-order total differential of the binary function, and then uses the mathematical induction method to derive the general formula for solving the high-order total differential of the n-variable function at the dimension level, but the derivation process is based on the condition that the n-order partial derivative is continuous. For the calculation of the higher order total differential formula of the ternary function and the higher order total differential formula of the multivariate function, when the function cannot be simplified by the mathematical induction method, the formula can be nested to realize the rapid calculation of the higher order total differential. However, for functions with discontinuous higher-order partial derivatives, other methods need to be used to solve them.

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