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Fuzzy Rule Interpolation Methods Based on Sparse Rule Bases

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ABSTRACT

Fuzzy rule interpolation algorithms have broad applications in computational fuzzy inference systems. This paper systematically introduces interpolation methods based on α-cuts. It focuses on two classical α-cut-based interpolation methods: the KH fuzzy rule interpolation method and the Lagrange fuzzy rule interpolation method. Through theoretical analysis and comparative studies, the fundamental principles, performance characteristics, and limitations of these two interpolation algorithms are explored in depth. Based on this, a fuzzy inference system for the "tip calculation problem" was constructed using the MATLAB Fuzzy Toolbox. Empirical studies were conducted to verify the necessity and feasibility of applying fuzzy rule interpolation techniques in sparse rule bases. The results indicate that fuzzy rule interpolation methods effectively enhance inference performance in sparse rule bases, providing essential theoretical foundations and practical support for the application of fuzzy inference systems.

KEYWORDS

Fuzzy rule interpolation; Fuzzy inference; Sparse rule base; Membership function

1. INTRODUCTION

Fuzzy set theory has been demonstrated to be an effective method for dealing with imprecision, uncertainty, and approximation in data and knowledge, thus providing solutions for complex real-world problems for which traditional Boolean logic is ineffective. Its wide range of applications extends to system control, data prediction, and expert systems [1, 2, 3]. Conventional fuzzy inference methods, such as the Compositional Rule of Inference (CRI), necessitate a dense fuzzy rule base [4], implying that for any given input, a specific fuzzy rule must be present within the rule base. However, in real-world applications, the procurement of a dense rule base is often impractical for various reasons [5, 6], including: (1) the construction of the rule base is often incomplete, encompassing both human expert knowledge and knowledge acquired through machine learning techniques. (2) the reduction of rule numbers is intentional to decrease system complexity, leading to a sparse or incomplete rule base. The existence of sparse rule bases complicates the direct implementation of traditional fuzzy inference methods, thus hindering their practical use.

To address this issue, fuzzy rule interpolation (FRI) techniques have been developed. These methods generate approximate inference results through interpolation, thus overcoming the dependency of traditional CRI methods on a dense rule base and enabling fuzzy inference in sparse rule bases. Since Kaczynski and Hirota first proposed the fuzzy rule interpolation method based on linear transformations (KH method), extensive research has led to numerous optimisations and innovations

[7, 8, 9]. Currently, FRI methods are mainly categorised into two types: α-cut-based interpolation methods and intermediate-rule-based interpolation methods.

The α-cut-based interpolation method (i.e., the non-transformation method) was one of the early research directions in fuzzy rule interpolation. This method involves the calculation of the distance measure between an unmatched input and adjacent rules, the generation of interpolation results using α-cuts, and the construction of the final fuzzy set based on the resolution principle. The KH method, proposed by Kaczynski and Hirota, is a notable approach within this category. However, it has been observed that the KH method may produce anomalous conclusions in some instances. To address this limitation, several improved methods have been introduced. For example, Péter Baranyi et al. proposed a novel approach to circumvent anomalous conclusions in the KH method [7] by constraining the number of characteristic points in fuzzy sets, thereby eradicating anomalies while preserving the computational simplicity and efficiency of the KH method. In a similar vein, Qiao et al. developed a similarity-based improvement to the KH method, ensuring that interpolation results always form convex fuzzy sets [9]. Additionally, Lu Zhengding, Wang Tianjiang, and Fu Haidong proposed a geometric similarity-based fuzzy rule interpolation method, which effectively guarantees the convexity and regularity of inference results [10]. Chang et al. introduced an interpolation method based on fuzzy set slopes, further enhancing inference performance under sparse rule bases [11]. In 2013, Chen et al. proposed a weighted multi-rule interpolation method based on polygonal membership functions, addressing deficiencies in traditional methods when handling complex rule bases [12]. In 2016, Chen and his team further introduced a weight-learning-based fuzzy interpolation inference method, which automatically learns the optimal weights of antecedent variables in rules, significantly improving inference accuracy and successfully applying it to truck rear-end control problems [13]. In recent years, Chen and Adam introduced two new adaptive fuzzy interpolation reasoning (AFIR) methods in 2017 [14] and 2018 [15], respectively, resolving inconsistencies that may arise in traditional inference processes. The validation of these methods was undertaken using a diarrheal disease prediction problem, a process which demonstrated their superiority. The experimental results indicate that these two AFIR methods outperform the approaches proposed by Yang and Shen [16] as well as Cheng et al. [17] in terms of inference result similarity.

In the context of the research above, the present paper systematically studies four α-cut-based fuzzy rule interpolation methods: the KH fuzzy rule interpolation method, the Lagrange fuzzy rule interpolation method, and the geometric parameter Lagrange fuzzy rule interpolation method. Through theoretical analysis and experimental comparisons, the mathematical principles, performance advantages, and limitations of each method are explored in detail. To further validate the practicality of the theoretical research, this paper uses the classical "tip calculation problem" as a case study. A fuzzy inference system is constructed utilising the MATLAB fuzzy toolbox, and experimental data substantiates the feasibility and efficacy of employing fuzzy rule interpolation techniques in sparse rule bases.

2. FUZZY RULE INTERPOLATION METHODS

For ease of discussion, the following concepts are defined:

Definition 1: Given a fuzzy set A_1, A_2 in the universe of discourse $X \times Y$, if $A_1 \cap A_2 = \varnothing$, then A_1, A_2 are said to be disjoint.

Definition 2: Given a rule base $R = \{A_i \to B_i \mid 0 < i \le n\}$, where A_i represents the antecedents of fuzzy rules, Suppose that $A_1 < A_2 < \cdots < A_i < A_{i+1} < \cdots < A_n$. If there exists at least one pair $0 < i < n$, such that A_i, A_{i+1} are disjoint, then the rule base R is considered sparse.

Definition 3: Given a rule base $R=\{A_i \rightarrow B_i \mid 0<i \leq n\}$, where A_i represents the antecedents of fuzzy rules, Suppose that $A_1<A_2<\cdots<A_i<A_{i+1}<\cdots<A_n$. If for any $0<i<n$, the intersection $A_i \cap A_{i+1} \neq \varnothing$, then the rule base R is considered dense.

Definition 4: Let $(L, \leq)$, $A \subseteq L$ be a partially ordered set. If $\forall \alpha \in A$, the condition $\alpha \leq \mathrm{r}$ holds, then r is called an upper bound of fuzzy set A. If r_0 is the smallest element among all upper bounds of A, then r_0 is called the least upper bound of fuzzy set A, denoted as $\mathrm{r}_0=\sup\{\alpha \mid \alpha \in A\}$.

Definition 5: Let $(L, \geq)$, $A \subseteq L$ be a partially ordered set. If $\forall \alpha \in A$, the condition $\alpha \geq \mathrm{r}$ holds, then r is called a lower bound of the fuzzy set A. If r_0 is the largest element among all lower bounds of A,, then r_0 is called the greatest lower bound of A, denoted as $\mathrm{r}_0=\inf\{\alpha \mid \alpha \in A\}$.

Definition 6: For a fuzzy set A, if there exists x_0, such that $\mu(x_0)=1$, then A is normal.

Definition 7: For A fuzzy set A, if for any $x, a<x<b$, such that $\mu(x) \geq \min(\mu_A(a), \mu_A(b))$, then A is convex.

Definition 8: Suppose that A_1, A_2 are fuzzy sets on the domain X, and $A_1 \prec A_2$. The upper bound distance between the cuts of $A_{1\alpha}$, $A_{2\alpha}$ is the difference between the upper bounds of the cuts, and the lower bound distance is the difference between the lower bounds of the cuts, where d represents the Euclidean distance. Specifically, the formulas are as follows:

$$\mathrm{d}_L(A_{1\alpha}, A_{2\alpha})=d(\inf\{A_{1\alpha}\}, \inf\{A_{2\alpha}\})=\inf\{A_{2\alpha}\}-\inf\{A_{1\alpha}\} \tag{1}$$

$$\mathrm{d}_U(A_{1\alpha}, A_{2\alpha})=d(\sup\{A_{1\alpha}\}, \sup\{A_{2\alpha}\})=\sup\{A_{2\alpha}\}-\sup\{A_{1\alpha}\} \tag{2}$$

Assuming triangular membership functions, the membership functions shown in Fig.1 (left) illustrate the membership functions of the fuzzy rule antecedents in the input domain X. In contrast, Fig.1 (right) demonstrates the membership functions of the fuzzy rule consequents B_1, B_2, B^* in the output domain Y. It is known that the fuzzy rule base contains rules $A_1 \Rightarrow B_2$, $A_2 \Rightarrow B_2$, and that the conditions in the domain do not overlap. When the input variable is A^*, and it satisfies the given conditions $A_1 \prec A^* \prec A_2$, the goal is to solve for the rule consequents B^* corresponding to A^*.

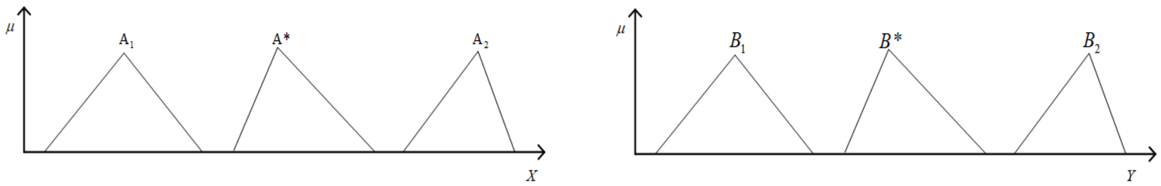

Figure 1. triangular fuzzy membership functions.

2.1. KH Fuzzy Rule Interpolation Method.

The KH fuzzy rule interpolation algorithm, as the inaugural published fuzzy rule interpolation method, is predicated on the approximate analogy reasoning concept that was introduced by Turksen [18]. The initial conception of the algorithm was to reduce the complexity of fuzzy rule bases [19]. However, its applicability is constrained by the interpretability of fuzzy theory [20]. The method must satisfy the following strict conditions: (1) The fuzzy sets in both the antecedents and consequents must be convex. (2) To simplify calculations, convex regular fuzzy sets (Convex Normal Fuzzy Set, CNF) must have bounded support and continuous membership functions. (3) There must be a clear partial order relationship between the convex regular fuzzy sets of each variable.

The KH fuzzy rule interpolation method calculates based on the fuzzy distance between the rule antecedents and the input facts, as well as the proportional relationship between the rule consequents and the corresponding conclusions. The linear interpolation calculation formulas are as follows:

$$\mathrm{d}_L(A_{1\alpha}, A_\alpha^*) : \mathrm{d}_L(A_\alpha^*, A_{2\alpha}) = \mathrm{d}_L(B_{1\alpha}, B_\alpha^*) : \mathrm{d}_L(B_\alpha^*, B_{2\alpha}) \tag{3}$$

$$\mathrm{d}_L(A_{1\alpha}, A_\alpha^*) : \mathrm{d}_L(A_\alpha^*, A_{2\alpha}) = \mathrm{d}_L(B_{1\alpha}, B_\alpha^*) : \mathrm{d}_L(B_\alpha^*, B_{2\alpha}) \tag{4}$$

According to Definition 8, the values of the rule consequents B^* are calculated as follows:

$$\min\{B_\alpha^*\} = \frac{\dfrac{\inf\{B_{1\alpha}\}}{d_L(A_\alpha^*, A_{1\alpha})} + \dfrac{\inf\{B_{2\alpha}\}}{d_L(A_\alpha^*, A_{2\alpha})}}{\dfrac{1}{d_L(A_\alpha^*, A_{1\alpha})} + \dfrac{1}{d_L(A_\alpha^*, A_{2\alpha})}} \tag{5}$$

$$\max\{B_\alpha^*\} = \frac{\dfrac{\sup\{B_{1\alpha}\}}{d_U(A_\alpha^*, A_{1\alpha})} + \dfrac{\sup\{B_{2\alpha}\}}{d_U(A_\alpha^*, A_{2\alpha})}}{\dfrac{1}{d_U(A_\alpha^*, A_{1\alpha})} + \dfrac{1}{d_U(A_\alpha^*, A_{2\alpha})}} \tag{6}$$

Finally, using the decomposition principle [21] of fuzzy sets interpolative reasoning, the fuzzy set B^* corresponding to the input fact A^* can be inferred.

The KH fuzzy rule interpolation method plays a vital role in different scenarios within fuzzy theory and is easy to implement in practical applications. Its stable approximation characteristics are considered an essential part of fuzzy theory and numerical analysis. However, the applicability of this method is constrained by its interpretability in fuzzy theory. It is only suitable for cases where the fuzzy sets are represented by triangular or trapezoidal membership functions [20]. Additionally, the inference results B^* obtained from the KH method may sometimes fail to satisfy regularity and convexity properties [22], which limits its application scope.

2.2. Lagrange Fuzzy Rule Interpolation Method.

The KH fuzzy rule interpolation method often fails to ensure the regularity and convexity of the inferred fuzzy set B^*, which limits its accuracy in some instances [23]. To address this problem, Y. Shi and M. Mizumoto proposed the Lagrange fuzzy rule interpolation method suitable for sparse fuzzy rule bases [24]. This method constructs a Lagrange mapping to achieve interpolation between input and output fuzzy sets.

When the input fuzzy set A^* satisfies specific conditions, its characteristic points can be extracted based on Definition 4 and Definition 5. As shown in Fig.1 when $\alpha = 1$, the characteristic points are determined as follows:

$$\begin{aligned} a_{11} &= \inf\{A_{1\alpha}\}, a_{13} = \sup\{A_{1\alpha}\} \\ a_1 &= \inf\{A_\alpha^*\}, a_3 = \sup\{A_\alpha^*\} \\ a_{21} &= \inf\{A_{2\alpha}\}, a_{23} = \sup\{A_{2\alpha}\} \end{aligned} \tag{7}$$

Similarly, when $\alpha = 0$, the characteristic points are given by:

$$
\begin{aligned}
a_{12} &= \inf\{A_{1\alpha}\} = \sup\{A_{1\alpha}\} = A_{1\alpha} \\
a_{2} &= \inf\{A_{\alpha}^{*}\} = \sup\{A_{\alpha}^{*}\} = A_{\alpha}^{*} \\
a_{22} &= \inf\{A_{2\alpha}\} = \sup\{A_{2\alpha}\} = A_{2\alpha}
\end{aligned} \tag{8}
$$

To ensure that the rule consequents retain the same triangular membership function type as B_1 and B_2, the α-cut representation of fuzzy sets A_1, A_2, B_1, B_2, A^* is formulated as follows:

$$
\begin{aligned}
A_{1\alpha} &= [\inf\{A_{1\alpha}\}, \sup\{A_{1\alpha}\}] = [(a_{i2}-a_{i1})\alpha + a_{i1}, -(a_{i3}-a_{i2})\alpha + a_{i3}] \\
&= [\alpha a_{i2} + (1-\alpha)a_{i1}, \alpha a_{i2} + (1-\alpha)a_{i3}] \\
B_{i\alpha} &= [\inf\{B_{i\alpha}\}, \sup\{B_{i\alpha}\}] = [(b_{i2}-b_{i1})\alpha + b_{i1}, -(b_{i3}-b_{i2})\alpha + b_{i3}] \\
&= [\alpha b_{i2} + (1-\alpha)b_{i1}, \alpha b_{i2} + (1-\alpha)b_{i3}] \\
A_{\alpha}^{*} &= [\inf\{A_{\alpha}^{*}\}, \sup\{A_{\alpha}^{*}\}] = [(a_{2}-a_{1})\alpha + a_{1}, -(a_{3}-a_{2})\alpha + a_{3}] \\
&= [\alpha a_{2} + (1-\alpha)a_{1}, \alpha a_{2} + (1-\alpha)a_{3}]
\end{aligned} \tag{9}
$$

Where $i = 1, 2$, $\alpha \in [0,1]$. The membership function of B^* is expressed as follows:

$$
\begin{aligned}
B_{\alpha}^{*} &= [\inf\{B_{\alpha}^{*}\}, \sup\{B_{\alpha}^{*}\}] = [(\mathrm{b}_{2}-b_{1})\alpha + b_{1}, -(b_{3}-b_{2})\alpha + b_{3}] \\
&= [\alpha \mathrm{b}_{2} + (1-\alpha)b_{1}, \alpha \mathrm{b}_{2} + (1-\alpha)b_{3}]
\end{aligned} \tag{10}
$$

Based on the properties of triangular membership functions, a Lagrange mapping is established between B^* and A^* using α-cuts:

$$
B_{\alpha}^{*} = F(A_{\alpha}^{*}) = [F(\inf\{A_{\alpha}^{*}\}), (\sup\{A_{\alpha}^{*}\})] \tag{11}
$$

Substituting Equation (9) for A^* and Equation (10) into Equation (11) yields:

$$
\begin{aligned}
\alpha \mathrm{b}_{2} + (1-\alpha)b_{1} &= F(\alpha a_{2} + (1-\alpha)a_{1}) \\
\alpha \mathrm{b}_{2} + (1-\alpha)b_{3} &= F(\alpha a_{2} + (1-\alpha)a_{3})
\end{aligned} \tag{12}
$$

For any $\alpha \in [0,1]$, Lagrange interpolation constructs a mapping from A_{α}^{*} to B_{α}^{*} with respect to the parameters a_1, a_2, a_3:

$$
\mathrm{b}_{j} = F(a_{j}) = \frac{a_{j} - a_{2j}}{a_{1j} - a_{2j}} \mathrm{b}_{1j} + \frac{a_{j} - a_{1j}}{a_{2j} - a_{1j}} \mathrm{b}_{2j} \tag{13}
$$

Since B^* consists of its upper and lower bounds defined by α-cuts, it can be further represented as:

$$
\begin{aligned}
\inf\{B_{\alpha}^{*}\} = {} & [\frac{a_{2} - a_{22}}{a_{12} - a_{22}} b_{12} + \frac{a_{2} - a_{12}}{a_{12} - a_{12}} b_{22} - \frac{a_{1} - a_{21}}{a_{11} - a_{21}} b_{11} - \frac{a_{1} - a_{11}}{a_{21} - a_{11}} b_{21}]\alpha \\
& + \frac{a_{1} - a_{21}}{a_{11} - a_{21}} b_{11} + \frac{a_{1} - a_{11}}{a_{21} - a_{11}} b_{21}
\end{aligned} \tag{14}
$$

$$\sup\{B_\alpha^*\} = [\frac{a_3 - a_{23}}{a_{13} - a_{23}} b_{13} + \frac{a_3 - a_{13}}{a_{23} - a_{13}} b_{23} - \frac{a_2 - a_{22}}{a_{12} - a_{22}} b_{13} - \frac{a_2 - a_{12}}{a_{22} - a_{12}} b_{21}]\alpha + \frac{a_3 - a_{23}}{a_{13} - a_{23}} b_{13} + \frac{a_3 - a_{13}}{a_{23} - a_{13}} b_{23} \tag{15}$$

This completes the derivation of the Lagrange-based fuzzy rule interpolation method. The primary advantage of this method lies in its ability to preserve the triangular membership function structure of the inferred fuzzy set B^*, ensuring consistency and interpretability. However, this approach has certain limitations: (1) the computational process is complex, especially when membership function parameters vary; (2) in some cases, the inferred fuzzy set B^* may not satisfy regularity and convexity constraints; (3) if no valid α-cut points exist in the membership function, the method fails to produce a meaningful inference result [25].

3. CASE STUDY ON THE INFERENCE PERFORMANCE OF FUZZY RULE INTERPOLATION METHODS

The "tip problem" is a classic example in fuzzy inference systems, addressing how customers intuitively determine the amount of gratuity based on service quality when dining at a restaurant. By establishing a relationship between service quality and tipping behaviour, the system takes service quality evaluations as input, processes them through fuzzy inference, and outputs the corresponding tip amount.

In this study, service quality (service) is defined over an input domain and represented by three triangular membership functions: "good", "medium", and "poor". Two fuzzy rules are given:

If the service quality is good, the tip amount is high.

If the service quality is poor, the tip amount is low.

The MATLAB Fuzzy Logic Toolbox (Fuzzy Logic Designer) is used to model the "tip problem." The input variable “service” is defined over the range [0, 10], with its membership functions listed in Table 1. The output variable “tip” is defined over the range [0, 1], where values closer to 1 indicate higher tip amounts. The membership functions of “tip” are presented in Table 2.

Table 1. Fuzzy Membership Functions of the Feature Variable "service"

service	poor	medium	good
Membership	[0, 2.5, 3]	[3.8, 4.2, 5.6]	[7.2, 8.3, 10]

Table 2. Fuzzy Membership Functions of the Feature Variable "tip"

tip	low	high
Membership	[0, 0.353, 0.48]	[0.65, 0.846, 1]

The fuzzy inference system incorporated the two predefined fuzzy rules to perform reasoning on input data. As shown in Fig.2, when the service quality input was 2.5, the system matched the first rule labelled as "poor" and inferred a tip amount of 0.278. When the service quality input was 8.2, the system matched the second rule labelled as "good" and inferred a tip amount of 0.832.

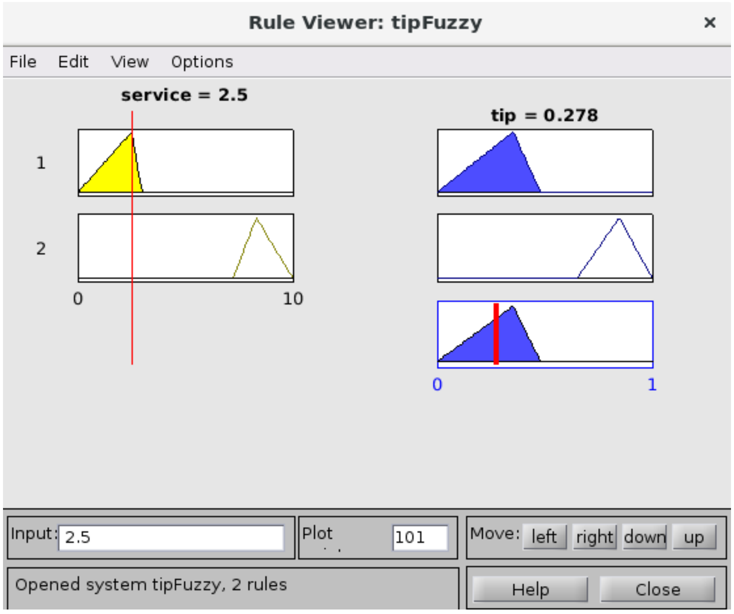

Figure 2. Fuzzy reasoning for non-sparse points

When the input falls on a sparse point (i.e., the "medium" membership function), for example, when the input service = 4.3, there are no matching rules in the fuzzy rule base, and the system cannot provide an inference result, as shown in Fig.3.

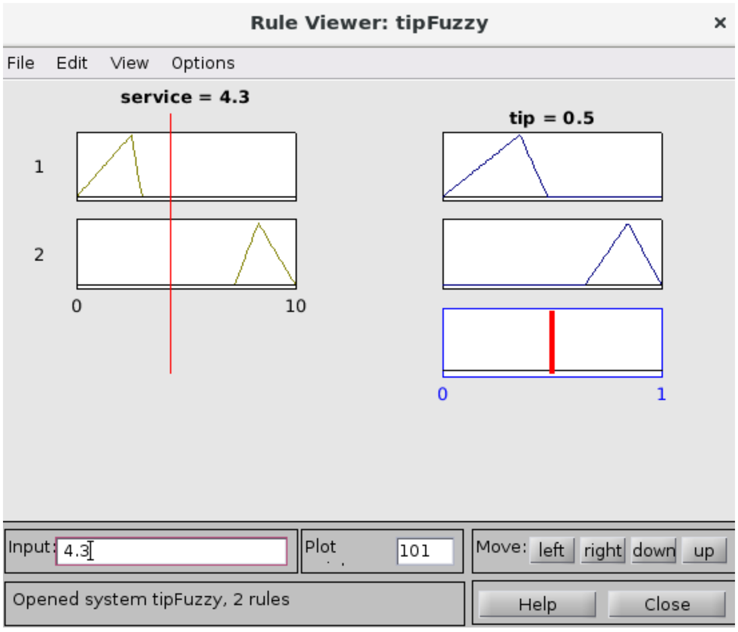

Figure 3. Fuzzy reasoning for sparse points

In order to address this issue, two interpolation methods, KH linear interpolation and fuzzy Lagrange interpolation, are employed to calculate the corresponding fuzzy rule consequents. The newly formulated rules are then integrated into the existing fuzzy system, thereby enhancing the overall rule base. The inference results obtained by the two interpolation methods are shown in Table 3.

Table 3. The rule consequents B^* obtained by the two fuzzy rule interpolation methods

Method	KH	Lagrange
B^*	(0.343, 0.497, 0.673)	(0.343, 0.497, 0.673)

Following the incorporation of the newly interpolated rules into the fuzzy inference system, the system is capable of handling inference issues at sparse points. When the input is service = 4.3, the inference results obtained using different interpolation methods are shown in Table 4.

Table 4. Inference results obtained by fuzzy rule interpolation algorithm

Method	KH	Lagrange
Service	4.3	4.3
Tip	0.503	0.523

This chapter validates the effectiveness of fuzzy rule interpolation techniques in sparse rule bases through the use of the so-called "tip problem" example. The experimental results demonstrate that fuzzy rule interpolation methods can effectively solve inference problems at sparse points, thereby preventing situations where the system fails to produce results. While there are slight differences in the inference results of various interpolation methods, all methods ensure the rationality and consistency of the results. In conclusion, fuzzy rule interpolation techniques are shown to hold significant practical value in the context of fuzzy inference with sparse rule bases, providing both reliable theoretical support and practical guidance for solving real-world problems.

4. SUMMARY

The fuzzy rule interpolation algorithm has extensive application value in computer-based fuzzy inference systems. This paper focuses on fuzzy rule interpolation inference algorithms under sparse rule bases, with a particular emphasis on the KH fuzzy rule interpolation method and the Lagrange fuzzy rule interpolation method. The theoretical foundations of these methods are thoroughly explored, and a systematic analysis of their advantages, disadvantages, and limitations is presented. Through the classic case study of the "tip problem," a fuzzy inference system is constructed to validate the necessity and feasibility of applying fuzzy rule interpolation techniques in sparse rule bases. The research results demonstrate that fuzzy rule interpolation methods can effectively enhance inference performance under sparse rule bases, providing significant theoretical support and practical guidance for the application of fuzzy inference systems.

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