

# Investigation of the Problem of Stabilisation of the Overturned Equilibrium Position of the Furuta Pendulum

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## ABSTRACT

Research goal: Investigation of the problem of stabilisation of the overturned equilibrium position of the Furuta Pendulum. Tasks to be solved in WRC / Research tasks: (1) Analytical review of incomplete-drive robots, their model representations and structural properties. (2) Analysis of kinematics and dynamic properties of incomplete-drive robots. (3) Research methods and control algorithms for incomplete-drive robots. (4) Synthesis of a control algorithm for stabilising Furuta's pendulum relative to an unstable overturned equilibrium position. Experimental modelling. (5) Analysis of dynamic properties and qualitative indicators of the obtained system. The aim of this final qualification work is to investigate incompletely driven systems, analyse their dynamic properties and control methods. Furuta's pendulum is investigated as an example of an incompletely driven system. In the work the model of the system is compiled, the control algorithm is synthesised, mathematical modelling in Simulink MATLAB environment is carried out.

## KEYWORDS

Pendulum on a trolley; Double pendulum; Furuta's Pendulum; Linear control; Non-linear control; Intelligent control

## 1. INTRODUCTION

During the last few decades, a number of academic, industrial and military developments have initiated research on analysing and creating control algorithms for various mechanical systems. Combining their efforts, engineers and scientists have developed several control system design methodologies that include linear control optimal control, adaptive and nonlinear control, and control methods that account for various system uncertainties. To date, control problems of systems in which the number of control actions is either equal to or greater than the number of degrees of freedom of the robotic device have been well studied. The problems of designing such structures and synthesising control algorithms for them are well studied and, among other things, provided with analytical methods. Despite the obvious advantages, the need for a large number of actuators to control such structures leads to increased requirements for the performance of actuators and accuracy of sensors, as well as increasing complexity of control, power consumption, weight and size, and the total cost of the robot. In recent years, however, there has been a growing interest in robotic systems in which the number of control actions is smaller than the number of generalised degrees of freedom, making them very dynamic and adaptable to different environments. These objects have been called "underactuated robotic systems" or "underactuated systems" [1]. The ability of such systems to perform complex tasks with minimal control actions has opened new opportunities in the development of various manipulators and mobile robotic systems. Such systems are energetically favourable due to the smaller number of energy consumers and, as a consequence, the mass and

dimensions are reduced. Also, with the reduction of the number of actuators, the system gains in cost. All-wheel drive systems help to solve quite a wide range of tasks. For example:

- Problems of stabilisation of vertical tilted equilibrium position of a pendulum link
- Problems of stabilisation of the centre of gravity of a walking mechanism -oscillatory motion optimisation problems
- trajectory control problems

Control of underdrive systems is a challenging task due to their nonlinear dynamics. Unlike all-wheel drive systems, these systems cannot be controlled using conventional control methods because direct linearisation is not feasible. Instead underdrive systems require the development of new control methods that take into account the passive dynamics of the system. The purpose of this final qualification work is to study incompletely driven systems, analyse their dynamic properties and control methods. Furuta's pendulum is studied as an example of an incompletely driven system. In the work the model of the system is compiled, the control algorithm is synthesised, mathematical modelling in Simulink MATLAB environment is carried out.

## **2. ANALYTICAL REVIEW OF PARTIALLY DRIVEN ROBOTS**

An underdriven system is a robot of non-trivial design. Underdrive can be caused by one of the following reasons [2]:

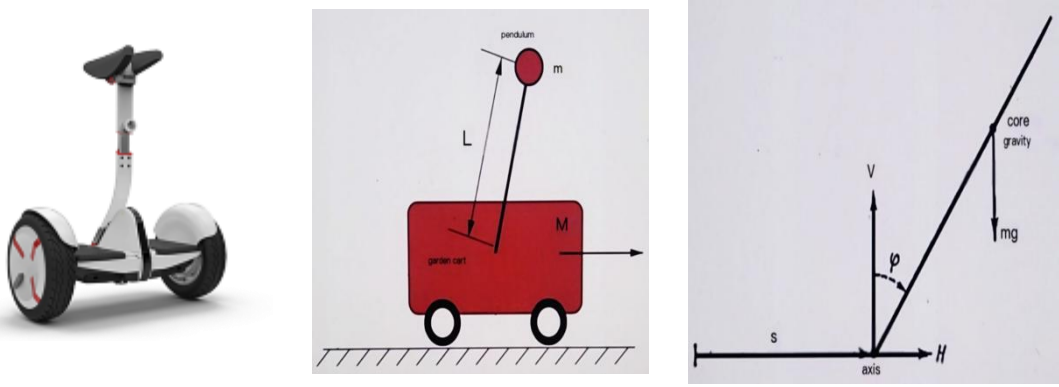
- the natural dynamics of the system. The list of such systems includes aeroplanes, helicopters, quadcopters and underwater vehicles.
- specially designed to reduce cost and weight. Examples include satellites with two motors and robots with flexible links.
- failure of one of the drives in the four-wheel drive system.

Partially driven systems have found their application in various fields of medicine, industry, logistics and transport. Prosthetics used in medicine using underdrive systems have the potential to significantly improve the quality of life of people with disabilities [3]. Autonomous vehicles capable of navigating in challenging environments with low energy consumption are also of great interest to the automotive industry [4]. Various research teams from around the world are working on the development of walking robots [5], aerial and underwater drones [6]. Also such a field of robotics as soft robotics, specialising in the construction of robots from soft materials similar to tissues of living organisms, has been actively developing recently [7].

Classical pendulum-like systems, such as the inverted pendulum on a cart [8], the two-link pendulum [9, 10], the Furuta pendulum [11], etc., are useful from a theoretical point of view, and act as benchmarks in the study of control algorithms for incompletely driven systems. Let us consider more details on the models of the above systems

### **2.1. Pendulum on a Trolley**

The inverted pendulum is a classic problem in dynamics and control theory and is widely used as a benchmark for testing control algorithms. The inverted pendulum on a trolley is the model that most closely describes the dynamics of a Segway -electric self-balancing scooter with two wheels. Its appearance with applied forces and displacements is shown in Figure 1.



**Figure 1.** Inverted pendulum on a trolley

An inverted pendulum is a pendulum whose centre of mass is above the fulcrum . Such a system is unstable, unlike a conventional pendulum, which is stable when suspended (it hangs down steadily under the influence of gravity). The stability of such a system can be ensured by a control system that controls the angular deflection of the pendulum and moves the fulcrum.of the pendulum and moves the fulcrum in the horizontal plane. In such a system, the pendulum axis (fulcrum) is mounted on a trolley that can move horizontally.

The trolley is driven by a small motor, which at time  $t$  applies a force  $\mu(t)$  to the trolley, which is the input variable , and the angular deflection of the pendulum is  $\varphi(t)$  . The mass of the pendulum is denoted by  $m$ ,  $L$  -is the distance between the axis and the centre of gravity,  $J$  is the moment of inertia about the centre of gravity,  $M$  is the mass of the cart . The pendulum is subjected to a force  $mg$  at the centre of gravity, as well as to horizontal  $H(t)$  and vertical  $V(t)$  reaction forces at the axis of the pendulum .

The following system of equations is valid for the system:

$$\begin{cases} m \frac{d^2}{dt^2}[s(t) + L \sin \varphi(t)] = H(t) \\ m \frac{d^2}{dt^2}[L \cos \varphi(t)] = V(t) - mg \\ J \frac{d^2 \varphi(t)}{dt^2} = LV(t) \sin \varphi(t) - LH(t) \cos \varphi(t) \\ M \frac{d^2 s(t)}{dt^2} = \mu(t) - H(t) - F \frac{ds(t)}{dt} \end{cases} \quad (1)$$

Friction is taken into account only when the bogie is moving; in the 4th equation in(1)  $F$  is the coefficient of friction. The friction of the pendulum axis is not taken into account. After transformations in (1)we have:

$$\begin{cases} m\ddot{s}(t) + mL\ddot{\varphi}(t)\cos\varphi(t) - mL\dot{\varphi}^2(t)\sin\varphi(t) = H(t) \\ -mL\ddot{\varphi}(t)\sin\varphi(t) - mL\dot{\varphi}^2(t)\cos\varphi(t) = V(t) - mg \\ J\ddot{\varphi}(t) = LV(t)\sin\varphi(t) - LH(t)\cos\varphi(t) \\ M\ddot{s}(t) = \mu(t) - H(t) - F\dot{s}(t) \end{cases} \quad (2)$$

In order to simplify the equations, it is assumed that the mass  $m$  is small compared to the mass  $M$  mass  $m$  is small compared to the mass  $M$ , so the horizontal reaction  $H(t)$  on the motion of the cart can be neglected. reaction  $H(t)$  to the motion of the cart. Considering the above and expressing  $H(t)$  and  $V(t)$  from the first two equations in (2):

$$\begin{cases} m\ddot{s}(t) + mL\ddot{\varphi}(t)\cos\varphi(t) - mL\dot{\varphi}^2(t)\sin\varphi(t) = H(t) \\ -mL\ddot{\varphi}(t)\sin\varphi(t) - mL\dot{\varphi}^2(t)\cos\varphi(t) = V(t) - mg \\ J\ddot{\varphi}(t) = LV(t)\sin\varphi(t) - LH(t)\cos\varphi(t) \\ M\dot{s}(t) = \mu(t) - H(t) - F\dot{s}(t) \end{cases} \quad (3)$$

Dividing by  $(J + mL^2)$  we obtain:

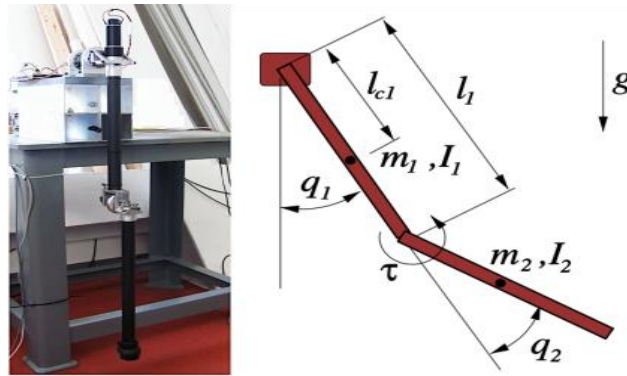
$$\begin{cases} \ddot{\varphi}(t) - \frac{g}{L'}\sin\varphi(t) + \frac{1}{L'}\dot{s}(t)\cos\varphi(t) = 0 \\ \ddot{s}(t) = \frac{1}{M}\mu(t) - \frac{F}{M}\dot{s}(t) \end{cases}, \quad (4)$$

Where  $L' = \frac{J+mL^2}{mL}$  is the effective length of the pendulum, since the motion of the of the mathematical pendulum of length  $L$  is also described by the first equation in (4).

The problem here is to stabilise the position of the pendulum in the vertical inverted state.

## 2.2. Double-arm Pendulum

A bipendulum is a pendulum with another pendulum attached to its end. The double pendulum is a simple physical system that exhibits a variety of dynamic behaviour with significant dependence on initial conditions. In this review, however, the double-link pendulum is considered as a robotic manipulator with two links, in which only the second link can be controlled. (there is an actuator only in the elbow). Such a system is called an Acrobot (from acrobatic robot). Its appearance and characteristic parameters are shown in Figure 2.



**Figure 2.** Two-link pendulum (Acrobot)

The standard equation for multi-link manipulators is:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = B(q)u \quad (5)$$

Where  $M(q)$  is the inertia matrix;  $C(q, \dot{q})$  is the matrix of Coriolis and centrifugal forces;  $G(q)$  - vector of gravitational forces,  $B(q) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ ;

$u = [0 \quad \tau]$  - moments of force, developed by the actuators at the shoulder and elbow joints;  $q = [\theta_1 \quad \theta_2]^T$ .

For this system, the above matrices are of the form:

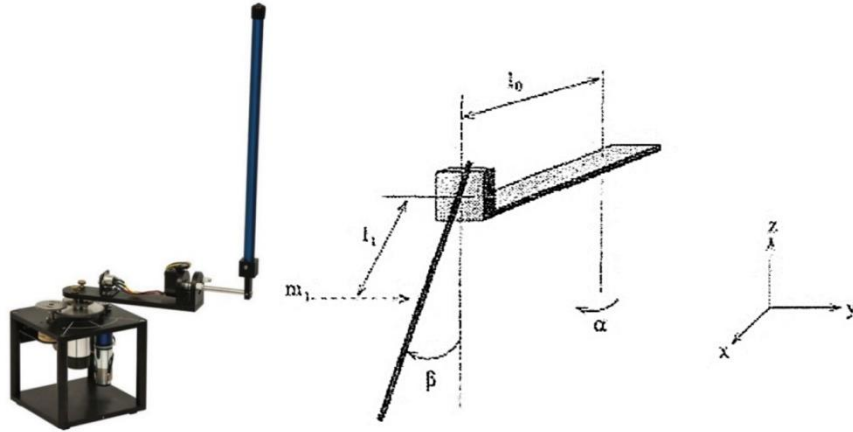
$$\begin{aligned}
M(q) &= \begin{bmatrix} I_1 + I_2 + m_2 l_1^2 + 2m_2 l_1 l_{c2} \cos \theta_2 & I_2 + m_2 l_1 l_{c2} \cos \theta_2 \\ I_2 + m_2 l_1 l_{c2} \cos \theta_2 & I_2 \end{bmatrix}, \\
C(q, \dot{q}) &= \begin{bmatrix} -2m_2 l_1 l_{c2} \sin \theta_2 \dot{q}_2 & -m_2 l_1 l_{c2} \sin \theta_2 \dot{q}_2 \\ m_2 l_1 l_{c2} \sin \theta_2 \dot{q}_1 & 0 \end{bmatrix}, \\
G(q) &= \begin{bmatrix} m_1 g l_{c1} \sin \theta_1 + m_2 g (l_1 \sin \theta_1 + l_{c2} \sin(\theta_1 + \theta_2)) \\ m_2 g l_{c2} \sin(\theta_1 + \theta_2) \end{bmatrix}
\end{aligned} \tag{6}$$

The resulting model is nonlinear. The task here is to stabilise the position of the pendulum in the vertical raised position.

### 2.3. Furuta's Pendulum

It is convenient to study real pendulum-like systems on prototypes, which include the Furuta pendulum. This pendulum is characterised by the presence of unstable and stable equilibrium points. Any small external influence takes the pendulum out of the unstable equilibrium position and brings it to the stable equilibrium position, in the vicinity of which weakly damped oscillations arise.

The Furuta pendulum or rotating inverted pendulum is a system consisting of an arm that is driven into rotation in the horizontal plane by an electric motor, and a lever that rotates in the vertical plane.



**Figure 3.** Furuta's Pendulum

The equations of motion of the pendulum are as follows:

$$\begin{bmatrix} (J_{z_0} + m_1 l_0^2 \sin^2 \beta) \ddot{\alpha} + m_1 l_1^2 \dot{\alpha} \dot{\beta} \sin \beta \cos \beta + m_1 l_0 l_1 \cos \beta \ddot{\beta} - \\ -m_1 l_0 l_1 \sin \beta \dot{\beta} + C_0 \dot{\alpha} + F_f \\ (J_{z_1} + m_1 l_1^2) \ddot{\beta} + m_1 l_0 l_1 \ddot{\alpha} \cos \beta - m_1 l_1^2 \dot{\alpha}^2 \sin \beta \cos \beta + \\ + m_1 g l_1 \sin \beta + C_1 \dot{\beta} \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}, \tag{7}$$

Where  $u$  is the torque developed by the motor,  $C_0 \dot{\alpha}$ . And  $C_1 \dot{\beta}$ . - are the forces of viscous friction in the corresponding joints,  $J_{z_0}$  is the moment of inertia of the arm,  $J_{z_1}$  is the moment of inertia of the lever,  $l_0$  is the length of the arm,  $l_1$  is the length of the lever from the joint to the centre of mass of the lever,  $m_1$  is the mass of the lever,  $F_f = K_f \cdot \text{sign}(\dot{\alpha})$  is the dry friction in the shoulder joint (in the joint between the motor shaft and the arm).

The sing function is approximated by a smooth function and is obtained:

$$F_f = K_f \cdot \sigma(\dot{\alpha}) = K_f \cdot \left(1 - \frac{2}{e^{2k\dot{\alpha}} + 1}\right) \quad (8)$$

The mathematical model of a pendulum can be written in the following form:

$$M \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix} + D \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} + K = U, \quad (9)$$

Where

$$\begin{aligned} M &= \begin{bmatrix} J_{z_0} + m_1 l_0^2 + m_1 l_1^2 \sin^2 \beta & m_1 l_0 l_1 \cos \beta \\ m_1 l_0 l_1 \cos \beta & J_{z_0} + m_1 l_1^2 \end{bmatrix}, \\ D &= \begin{bmatrix} C_0 + m_1 l_1^2 \dot{\beta} \sin \beta \cos \beta & m_1 l_1^2 \dot{\alpha} \sin \beta \cos \beta - m_1 l_0 l_1 \dot{\beta} \sin \beta \\ -m_1 l_1^2 \dot{\alpha} \sin \beta \cos \beta & C_1 \end{bmatrix}, \\ K &= \begin{bmatrix} K_f \cdot \sigma(\dot{\alpha}) \\ m_1 g l_1 \sin \beta \end{bmatrix}, U = \begin{bmatrix} u \\ 0 \end{bmatrix}. \end{aligned} \quad (10)$$

The resulting model is nonlinear.

### 3. RESEARCH METHODS AND ALGORITHMS FOR THE CONTROL OF INCOMPLETELY DRIVEN ROBOTS

Control of incompletely actuated systems aims at finding a control law with such feedback that stabilizes the system when the system is not fully actuated. control law with such feedback that stabilizes the system in the presence of various uncertainties and external perturbations. presence of various uncertainties and external perturbations. The reason of the difficulty in finding a control algorithm is that the number of actuators is smaller than the number of degrees of freedom to be controlled, and many traditional nonlinear methods of control are not directly applicable. In recent years, many control methods have been developed many control methods that address this problem. These methods include. include methods based on 1) linearization using a linear- quadratic controller [9]; 2) partial feedback linearization [12]; 3) backstepping [13]; 4) sliding mode [14]; 5) energy-based approach [15]; 6) passification method [16]; 7) fuzzy logic [17]. In this section.some of these popular control methods are reviewed.

#### 3.1. Linear Control

Linearization using linear-quadratic controller the linear-quadratic controller is the most importanttools in optimal control theory today. C

This approach is used to solve the problem of local stabilization in incompletely driven systems. incompletely driven systems. The method allows linearization, by expanding the nonlinear functions into a Taylor series and replacing them with the first terms of the of the expansion [9]. Let us consider the method in more detail.

A nonlinear system of the form:

$$\dot{x} = f(x, u) \quad (11)$$

And a stabilizable equilibrium point  $(x_0, u_0)$  such that  $f(x_0, u_0) = 0$ . Let us introduce a new coordinate system:

$$\bar{x} = x - x_0, \quad \bar{u} = u - u_0, \quad (12)$$

And consider the system:

$$\dot{\bar{x}} = \dot{x} = f(x, u), \quad (13)$$

Which can be decomposed into a Taylor series, resulting in a linearized system:

$$\dot{\bar{x}} \approx f(x_0, u_0) + \frac{\partial f(x_0, u_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, u_0)}{\partial u}(u - u_0) = A\bar{x} + B\bar{u}. \quad (14)$$

Now we can compose a stabilizing regulator for such a system, using some optimality criterion (in this case, a quadratic functional):

$$J(x_0) = \int_0^{\infty} [x^T(t)Qx(t) + u^T(t)Ru(t)] dt, \quad (15)$$

$$x(0) = x_0, \quad Q = Q^T > 0, \quad R = R^T > 0.$$

The negative feedback control law should minimize this criterion. Such control has the form:

$$\bar{u} = -R^{-1}B^T Px = -K\bar{x}$$

$$u = u_0 - K(x - x_0), \quad (16)$$

Where P is found from the solution of the Riccati equation:

$$-\dot{P} = PA + A^T P - PBR^{-1}B^T P + Q. \quad (17)$$

It should be noted that, in spite of its optimality in a certain parameter for a given system, such a regulator is not necessarily the best solution for maximizing the convergence region of the point in the vicinity of which the system has been linearized.

## 3.2. Nonlinear Control

### 3.2.1. Partial feedback linearization

In all incompletely driven systems, there is a possibility of partial linearization by feedback. This property was discovered by Spong [18], and is a consequence of the positive definiteness of the inertia matrix. This linearization can be performed on both the controlled and uncontrolled links.

Let us consider again the model of an incompletely driven system in general form:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = F(q)\tau \quad (18)$$

Where  $M(q)$  is the inertia matrix;  $C(q, \dot{q})$  is the matrix of Coriolis and centrifugal forces;  $G(q)$  is the vector of gravitational forces;  $\tau \in \mathbb{R}^m$  is the control,  $F(q) \in \mathbb{R}^{n \times m}$  is the matrix of external forces,  $m < n$ ,  $q \in Q$  is the configuration vector belonging to the  $n$ -dimensional configuration space.

Suppose that  $F(q) = [0, I_m]^T$ , and we decompose the configuration vector as follows:  $q = (q_1, q_2) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ , where  $q_1$  represents the driverless configuration vector (uncontrolled links) and  $q_2$  represents the driven configuration vector (controlled links). Because of the introduced assumptions, the model of the underdriven system (18) will take the form:

$$\begin{bmatrix} m_{11}(q) & m_{12}(q) \\ m_{21}(q) & m_{22}(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} N_1(q, \dot{q}) \\ N_2(q, \dot{q}) \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix} \quad (19)$$

Where  $N_i(q, \dot{q})$  contains centrifugal, Coriolis and gravitational components.

Due to the absence of the controlling influence in the first equation in (19) it is not possible to linearise this system completely. However, it is possible to partially linearise this system. Moreover, such linearisation is possible both for the controlled link and for the uncontrolled link.

In the case of partial linearisation on the controlled link, there exists such control that:

$$\tau = \alpha(q)u + \beta(q, \dot{q}) \quad (20)$$

Which partially linearises the dynamics (19). As a result, we obtain:

$$\begin{aligned} \dot{q}_1 &= p_1 \\ \dot{p}_1 &= f_0(q, p) + g_0(q)u \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u \end{aligned} \quad (21)$$

Where  $\alpha(q)$  is a symmetric positively defined matrix, and

$$g_0(q) = -m_{11}^{-1}(q)m_{12}(q), f_0(q, p) = -m_{11}^{-1}(q)N_1(q, \dot{q}).$$

Partial linearisation along the uncontrolled link is possible only if the number of control actions is greater than or equal to the number of uncontrolled links. Let us consider such an incompletely driven system:

$$\begin{bmatrix} m_{00}(q) & m_{01}(q) & m_{02}(q) \\ m_{10}(q) & m_{11}(q) & m_{12}(q) \\ m_{20}(q) & m_{21}(q) & m_{22}(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_0 \\ \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} N_0(q, \dot{q}) \\ N_1(q, \dot{q}) \\ N_2(q, \dot{q}) \end{bmatrix} = \begin{bmatrix} \tau_0 \\ \tau_1 \\ 0 \end{bmatrix} \quad (22)$$

Where  $q = (q_0, q_1, q_2) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  c  $n_1 = n_2 = m, n_0 = n - 2m \geq 0$ .

In the case of partial linearisation over an unguided link, there exists a control such that:

$$\tau = \alpha_1(q)u + \beta_1(q, \dot{q}), \quad (23)$$

Which partially linearises the dynamics (22) on the set  $U = \{q \in \mathbb{R}^n / \det(m_{21}(q)) \neq 0\}$ . The result is:

$$\begin{aligned}
\dot{q}_0 &= p_0 \\
\dot{p}_0 &= u_0 \\
\dot{q}_1 &= p_1 \\
p_1 &= f_0(q, p) + g_0(q)u_0 + g_2(q)u_2 \\
\dot{q}_2 &= p_2 \\
\dot{p}_2 &= u
\end{aligned} \tag{24}$$

Where

$$\tau = (\tau_0, \tau_1), u = (u_0, u_1), u_1 = \alpha_0(q)u_0 + \alpha_2(q)u_2 + \beta_2(q, \dot{q}), f_0(q, p) = -m_{21}^{-1}(q)N_2(q, \dot{q}), g_0(q) = -m_{21}^{-1}(q)m_{20}(q), g_2(q) = -m_{21}^{-1}(q)m_{22}(q)$$

Due to the simplicity of implementation, this linearisation method is the most widely used.

### 3.2.2. Energy approach

For tasks such as controlling an inverted pendulum or a two-link pendulum, a link cannot be stabilised in an inverted position while it is near the horizontal position. It must first be swung so that it overcomes the 90° limit, only then stabilising its vertical equilibrium position. Therefore, two control loops are synthesised, i.e. two controllers: one for rocking and one for local asymptotic stability at the equilibrium point. The so-called energy approach has found its application in such a problem.

The algorithm for solving the problem of stabilising the equilibrium position is usually as follows. First, the partial feedback linearisation described above is used to simplify the dynamics of the system, then an energy approach to sway the link to the vertical position, then some other methods are applied to stabilise the link at the equilibrium point.

Let us consider the application of the energetic approach proposed by Spaughn [19], [20] on the example of the two-link acrobot-type pendulum described in the previous section. Let us consider again the system describing the dynamics of the pendulum [21]:

$$\begin{cases} M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + C_1 + \mu_1\dot{q}_1 + D_1\text{sign}(\dot{q}_1) = 0, \\ M_{21}\ddot{q}_1 + M_{22}\ddot{q}_2 + C_2 + \mu_2\dot{q}_2 + D_2\text{sign}(\dot{q}_2) = \tau, \end{cases} \tag{25}$$

Where  $M_{ij}(q)$  are moments of inertia;  $C_i(q, \dot{q})$  are Coriolis and centrifugal forces;  $\tau$  is control,  $\mu_i$  are viscous friction coefficients,  $D_i$  are kinetic friction forces.

Before applying the energy approach, we apply a partial feedback linearisation. As a result, we obtain the following control law:

$$\begin{aligned}
\tau = \frac{M_{11}M_{22} - M_{12}M_{21}}{M_{11}}u - \frac{M_{21}}{M_{11}}(C_1 + \mu_1\dot{q}_1 + D_1\text{sign}(\dot{q}_1)) + \\
+ C_2 + \mu_2\dot{q}_2 + D_2\text{sign}(\dot{q}_2).
\end{aligned} \tag{26}$$

Substituting the obtained control into the system (25), we obtain:

$$\begin{cases} M_{11}\ddot{q}_1 + M_{12}\ddot{q}_2 + C_1 + \mu_1\dot{q}_1 + D_1\text{sign}(\dot{q}_1) = 0, \\ \ddot{q}_2 = u. \end{cases} \tag{27}$$

Now it is possible to choose a control for the outer loop such that, the articulation  $q_2$  will swing in a given direction  $q_2^d$ . Let's set trajectory in  $q_2^d$  the form:

$$q_2^d = \alpha \arctan(\dot{q}_1). \quad (28)$$

Let's choose a control in the following form:

$$u = k_p (q_2^d - q_2) - k_d \dot{q}_2, \quad (29)$$

Where  $k_p$ ,  $k_d$  are positive coefficients. To achieve swing control of the acrobot, we apply a constant torque to the input constant torque from the initial state  $(q_1, q_2, \dot{q}_1, \dot{q}_2) = (0, 0, 0, 0)$  for the first few seconds, and then apply the control law, given by (26), (28) and (29).

Next, the control law of the inner loop is synthesised, stabilising the pendulum in position  $(q_1, q_2, \dot{q}_1, \dot{q}_2) = (\pi, 0, 0, 0)$ .

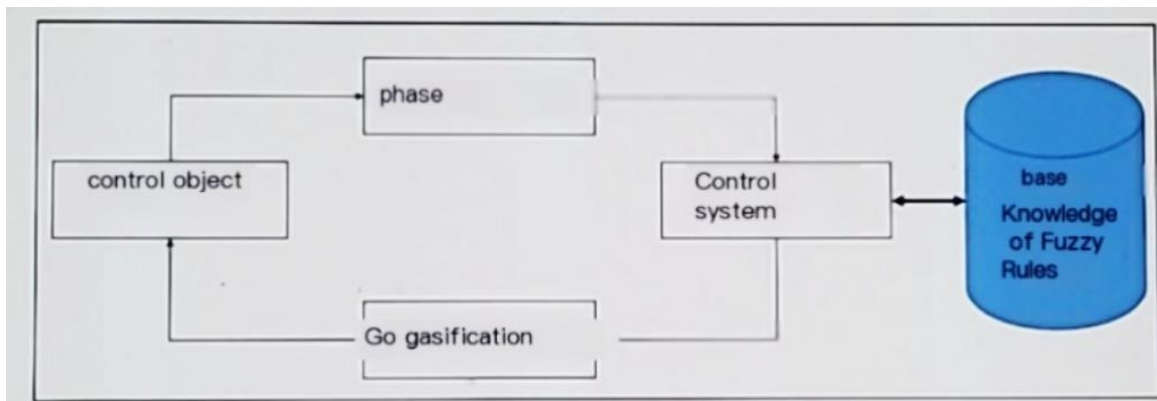
### 3.3. Intelligent Control

Despite continuous active research on the control of underdrive systems over the last decades, key technical problems such as reliability improvement, compensation of external disturbances and uncertainties are still relevant today. Developers have set out to make available control methods more adaptive and intelligent. As a result, neural networks and fuzzy logic have been applied to control tasks of incomplete drive systems.

#### 3.3.1. Fuzzy logic

In modern control theory, in addition to traditional methods based on strict mathematical formalisation, heuristic approaches have been developed and successfully applied, demonstrating a fairly high efficiency in practice. Methods using fuzzy logic are based on the formal representation of human knowledge about the object and, most importantly, the empirical experience of solving management problems, summarised by the expert in the concepts of 'better', 'worse', 'more often', 'less often' and his ideas about the 'optimality' of the management process [22]. The main feature that distinguishes this method from classical regulators is a more complex structure, non-linear nature of the dependence of the output variable on the input variable, multiple repetition of maximisation and minimisation operations in the algorithm, etc. In general, the control system based on fuzzy logic functions according to the following scheme [23] (Figure4):

- at the phasification stage, the transformation of crisp quantities measured at the output of the control object into fuzzy quantities described by linguistic variables in the knowledge base is performed;
- the control system (or regulator) uses fuzzy conditional rules contained in the knowledge base to transform fuzzy input data into the required fuzzy control actions;
- in the defuzzification stage, the fuzzy data are converted from the output of the controller into fuzzy values that are used to control the object.



**Figure 4.** Scheme of realisation of fuzzy control

The main development has been in fuzzy models for describing complex systems proposed by Tagaki-Sugeno [24] and Mamdani [25].

As examples of the use of methods based on fuzzy logic, the following works can be cited. In [26] they solve the problem of trajectory control of an incompletely driven autonomous underwater vehicle. In [27], a fuzzy controller is developed to control a two-link Pendubot pendulum to maintain periodic oscillations of the first link, while the second link remains in the upright position. Based on the inverted pendulum system on a trolley, a new type of laboratory bench, called trolley-pendulum-swing system, is proposed in [28]. In [29], an algorithm for trajectory control of an incompletely driven system with a ball on a plane based on fuzzy logic is proposed. The authors of [30] proposed a solution to the problem of stabilising the equilibrium position of an inverted pendulum on a cart.

However, fuzzy logic based methods have their own disadvantages in the form of low control accuracy.

### 3.3.2. Neural Networks

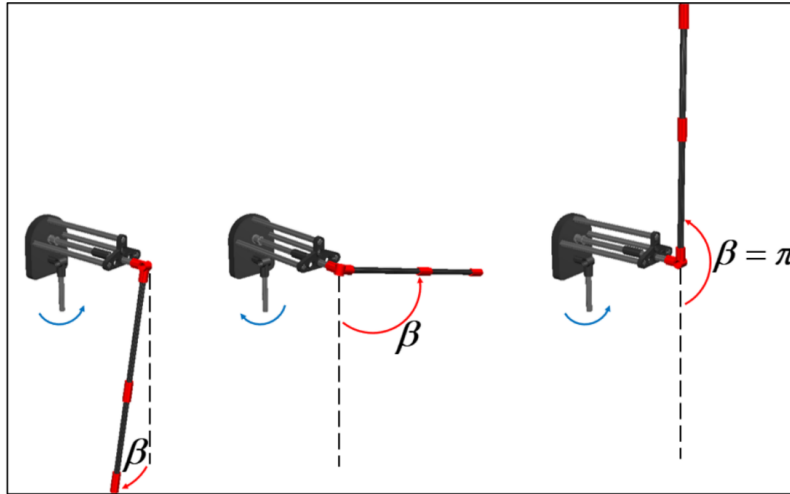
The structure of neural networks is inspired by the observed processes in natural networks of brain neurons. The learning process is carried out by adjusting the weights that represent the strength of interconnectivity of neurons based on certain learning algorithms. Neural networks have inherent learning ability and are able to approximate a nonlinear continuous function with arbitrary accuracy. Thus, there has been an active growth in the number of studies on the use of neural network approach for controlling incompletely driven robots. The following works can be cited as examples of the use of methods based on the application of neural networks. In [31], a neural network approach is applied neural network approach in the problem of controlling an inverted pendulum on a trolley. The underdriven system is represented as two subsystems. The first subsystem is fully controllable and is a second order active system describing translational motion and angular yaw motion. The second subsystem is a passive incomplete first-order system describing the pendulum motion, which is indirectly controlled through dynamic coupling with the active subsystem. Adaptive neural network control is used to control the motion of the active subsystem, i.e., the trolley itself. In [32], in order to maintain the equilibrium position of a two-link pendulum, a method of control method based on the energy approach with integration of the neural network approach used to compensate the effect of dynamic friction of the system.

## 4. PROBLEM STATEMENT

The Furuta pendulum, also known as a pivoting inverted pendulum, is considered as an underdriven system. This partially driven system is a mechanism with two degrees of freedom and two rotational joints. Essentially, the device consists of three elements: a motor and two rods called the lever and the pendulum. The motor shaft is connected to one end of the lever, causing angular movement of the lever in the horizontal plane, while the pendulum is connected to the free end of the lever through a link that can move freely and allows the pendulum to rotate in the vertical plane.

The control problem is as follows: we need to keep Furuta's pendulum in the vertical equilibrium position  $\beta = \pi$

The difficulty in constructing a suitable control law is that the pendulum cannot be stabilised in an inverted position as long as it is near the horizontal position  $\beta = -\pi + \pi k, k \in \mathbb{Z}$  (Рисунок 5).



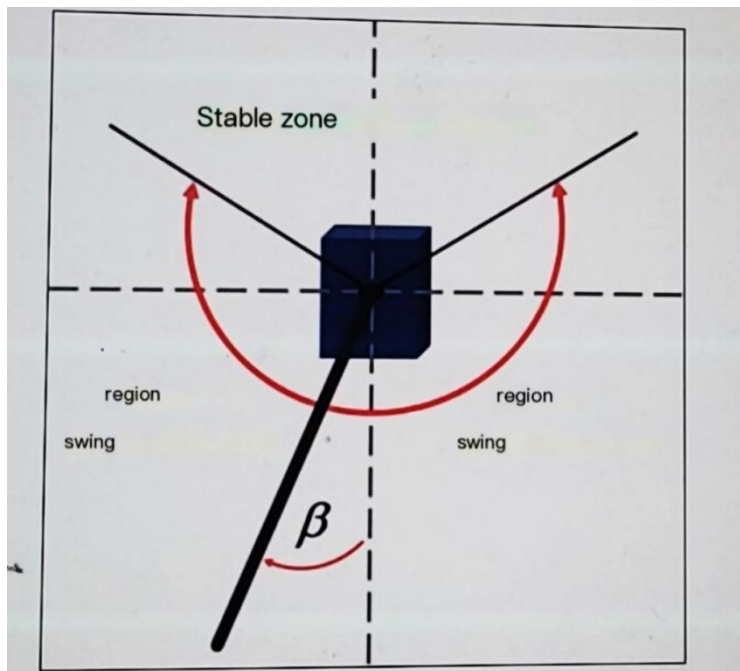
**Figure 5.** The process of swinging the Furuta pendulum

The pendulum must first be swung so that it crosses the  $90^\circ$ , threshold, only then stabilise its vertical equilibrium position. This raises the problem of synthesising two regulators. One is needed to swing the pendulum and the other to keep it in the equilibrium position.

## 5. SOLVING THE PROBLEM OF STABILISATION OF THE FURUTA PENDULUM

### 5.1. Synthesis of the Control Algorithm for Stabilisation Relative to the Unstable Overturned Equilibrium Position

First of all, let us synthesise the controller, which will allow us to swing the pendulum to the stabilisation region where  $\cos(\beta) < 0$  (Figure 6).



**Figure 6.** Schematic representation of control areas

There are different approaches to realise the swing process. In the framework of this problem we will use the so-called energy approach, which was described in the previous section. The kinetic energy of the lever rotation without taking into account the arm movement can be written in the form:

$$E = \frac{1}{2} (J_{z_1} + m_1 l_1^2) \dot{\beta}^2 - m_1 g l_1 (\cos \beta + 1). \quad (30)$$

The kinetic energy is zero when the pendulum is in the vertical up position  $\beta = \pi$ ,  $\dot{\beta} = 0$  and  $-2m_1 g l_1$  – in the vertical down position  $\beta = 0$ ,  $\dot{\beta} = 0$ . The pendulum should move along such a trajectory that its kinetic energy tends to zero. This is achievable by controlling the angular acceleration of the lever. The control law holds:

$$s = n g \cdot \text{sign}(E) \dot{\beta} \cos \beta, \quad (31)$$

Where  $n$  is the value on which the number of swings depends.

The desired acceleration is provided by a motor whose shaft is rigidly connected to the arm. The angular acceleration of the arm is found from the following expression:

$$\ddot{\alpha} = \frac{s}{l_0} = \frac{n g \cdot \text{sign}(E) \dot{\beta} \cos \beta}{l_0}, \quad (32)$$

Since the control signal  $v$  is the motor torque, the following conversion must be performed:

$$\ddot{\alpha} = v. \quad (33)$$

Next, it is necessary to perform feedback linearisation, which has already been described in the previous section, but is also still presented in [33], [34].

The next step is to control the rotation of the lever, stabilising it in the equilibrium position. The output  $y$  of the stabilising controller should be the angle  $\beta$ . Let us rewrite (9) in the following form:

$$\frac{d}{dt} \begin{bmatrix} \alpha \\ \dot{\alpha} \\ \beta \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \dot{\alpha} \\ \frac{1}{\det(M)} (a + (J_{z_1} + m_1 l_1^2) u - b) \\ \dot{\beta} \\ \frac{1}{\det(M)} (c - (m_1 l_0 l_1 \cos \beta) u + d) \end{bmatrix}, \quad y = \beta, \quad (34)$$

Where

$$a = -(J_{z_1} + m_1 l_1^2) (C_0 \dot{\alpha} + 2m_1 l_1^2 \dot{\alpha} \dot{\beta} \sin \beta \cos \beta - m_1 l_0 l_1 \dot{\beta}^2 \sin \beta + K_f \sigma(\dot{\alpha})),$$

$$b = m_1 l_0 l_1 \cos \beta (m_1 l_1 \dot{\alpha}^2 \sin \beta \cos \beta - C_1 \dot{\beta} - m_1 g l_1 \sin \beta),$$

$$c = m_1 l_0 l_1 \cos \beta (C_0 \dot{\alpha} + 2m_1 l_1^2 \dot{\alpha} \dot{\beta} \sin \beta \cos \beta - m_1 l_0 l_1 \dot{\beta}^2 \sin \beta + K_f \sigma(\dot{\alpha})),$$

$$d = -(J_{z_0} + m_1 l_0^2 + m_1 l_1^2 \sin^2 \beta) (m_1 l_1 \dot{\alpha}^2 \sin \beta \cos \beta - C_1 \dot{\beta} - m_1 g l_1 \sin \beta).$$

Control first appears in the second time derivative of the output  $y$ :

$$\ddot{y} = \frac{1}{\det(M)} (c - (m_1 l_0 l_1 \cos \beta) u + d). \quad (35)$$

Hence, we have a second order system, provided that  $\cos \beta \neq 0$ . Now the control law using feedback linearisation can be represented as:

$$u = \frac{\det(M)v - c - d}{-m_1 l_0 l_1 \cos \beta}, \beta \neq \frac{\pi}{2} + \pi k, k \in Z. \quad (36)$$

The system now has the form:

$$\ddot{y} = v. \quad (37)$$

Let us introduce a control law, which is represented as a PD controller

$$v = PD(\beta, \dot{\beta}) = -a_0(\beta - \pi) - a_1 \dot{\beta}, \quad a_0, a_1 > 0. \quad (38)$$

Now after swinging the systems should stabilise the lever in the upper vertical position. Subsystem:

$$\frac{d}{dt} \begin{bmatrix} \beta \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \dot{\beta} \\ v \end{bmatrix}. \quad (39)$$

Tabilised for (38), we can now determine the zero-dynamics conditions for (34).

Substituting  $\beta = \pi, \dot{\beta} = 0, v = 0$  into the original system (34) and (36), we obtain:

$$\begin{aligned} a &= -(J_{z1} + m_1 l_1^2)(C_0 \dot{\alpha} + K_f \sigma(\dot{\alpha})), \\ b &= 0, \\ c &= m_1 l_0 l_1 (C_0 \dot{\alpha} + K_f \sigma(\dot{\alpha})), \\ d &= 0. \end{aligned} \quad (40)$$

Hence we get the system:

$$\frac{d}{dt} \begin{bmatrix} \alpha \\ \dot{\alpha} \\ \beta \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \dot{\alpha} \\ \frac{1}{\det(M)}(J_{z1} + m_1 l_1^2)(-C_0 \dot{\alpha} - K_f \sigma(\dot{\alpha}) + u) \\ \dot{\beta} \\ \frac{1}{\det(M)}(m_1 l_0 l_1 (C_0 \dot{\alpha} + K_f \sigma(\dot{\alpha}) - u)) \end{bmatrix}. \quad (41)$$

To realise the condition  $\ddot{\beta} = v = 0$ , we introduce a control:

$$u = C_0 \dot{\alpha} + K_f \sigma(\dot{\alpha}). \quad (42)$$

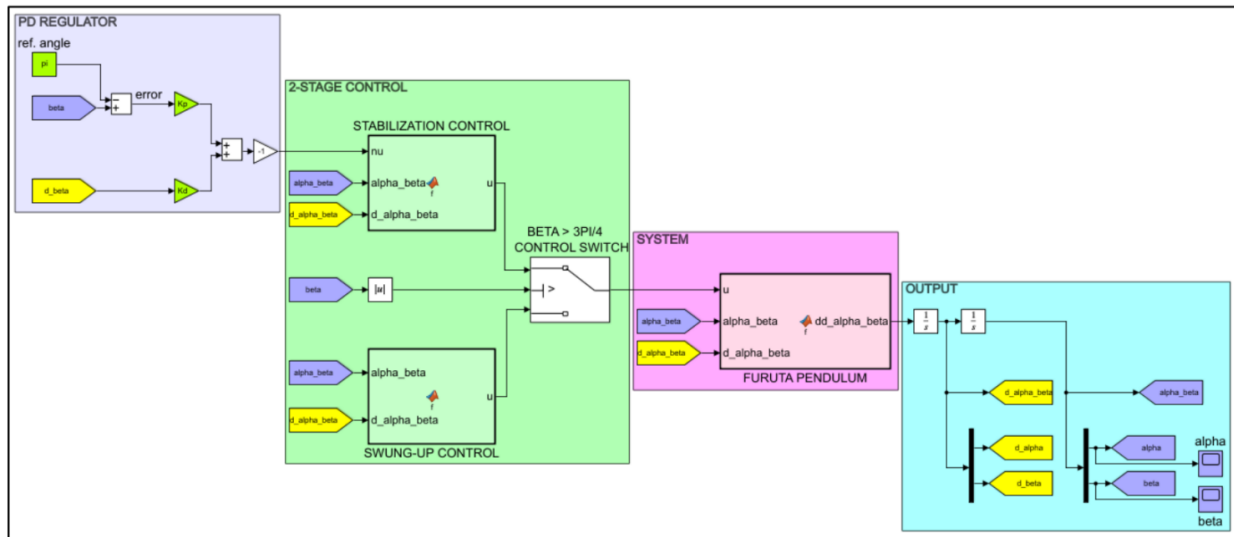
Substituting (42) into (41), we obtain the final system for the lever stabilised in he upper vertical position:

$$\frac{d}{dt} \begin{bmatrix} \alpha \\ \dot{\alpha} \\ \beta \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \dot{\alpha} \\ 0 \\ \dot{\beta} \\ 0 \end{bmatrix}. \quad (43)$$

## 5.2. Experimental Modelling

After the mathematical model of the system has been compiled and the control law has been synthesised. control law, allowing first to swing the pendulum to the stabilisation region and then to keep the equilibrium position, it is possible to make a model in Simulink MATLAB environment and carry out simulation to make sure that the obtained results are correct.

The modelling scheme of the pendulum control system Furuta is presented according to figure 7.

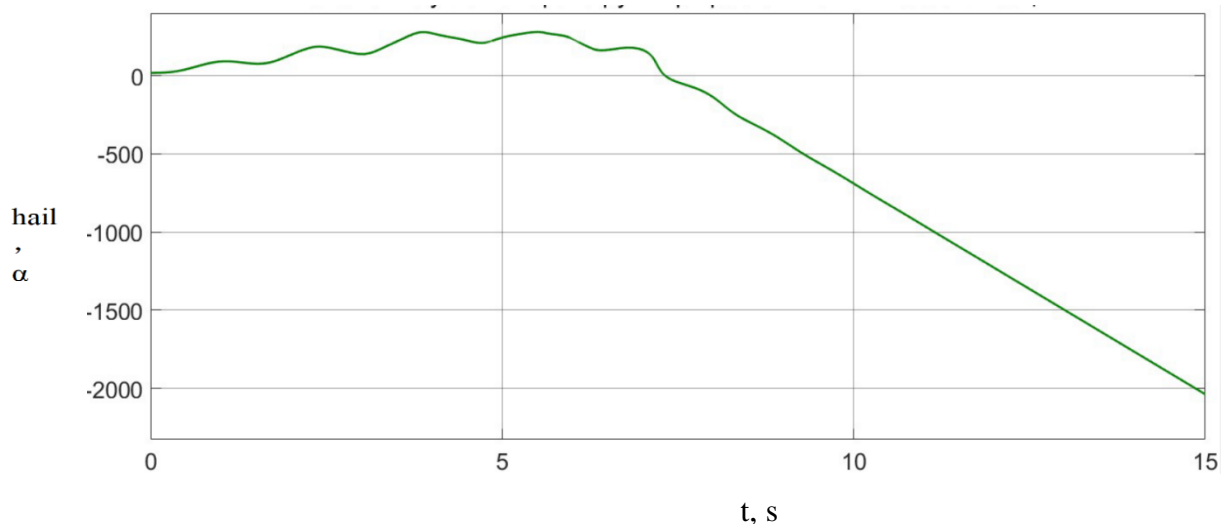


**Figure 7.** Modelling scheme

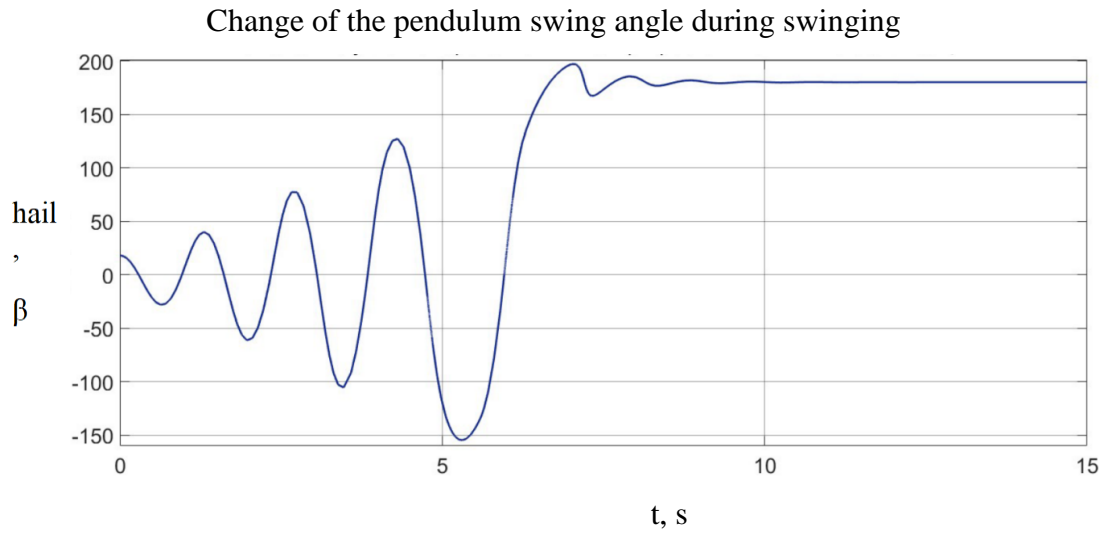
The control scheme is a two-stage controller. The first controller swings the pendulum, and when the value of the angle  $\beta$  becomes greater than  $135^\circ$ , the control system switches to the second controller which stabilises the pendulum position at  $\beta = \pi$ . Appendices A-B are program listings for the blocks responsible for the regulators and the control system itself.

The results of the simulation of the application of the regulator to the swing of the pendulum are given according to Figures 8-11.

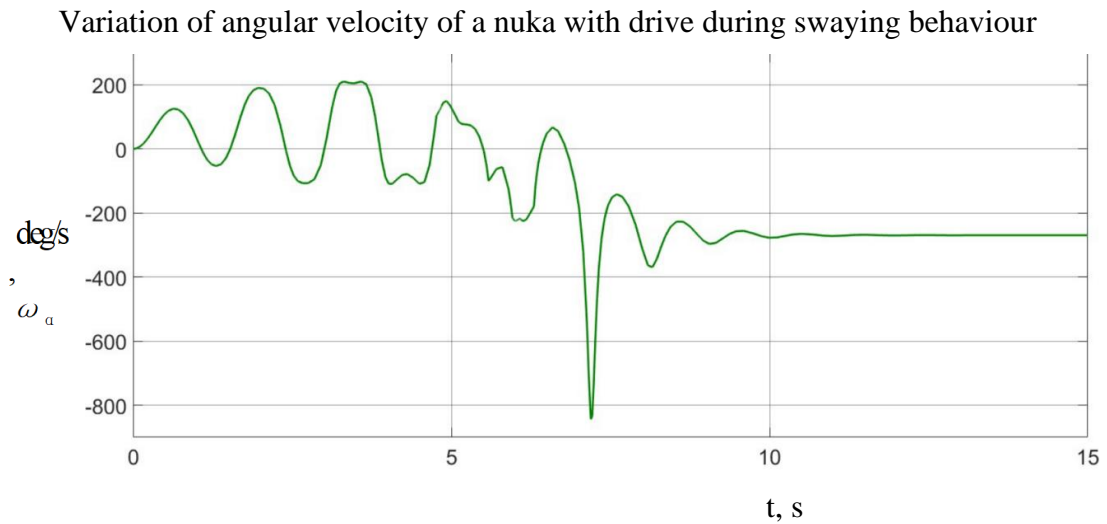
Changing the angle of arm swing and stabilisation



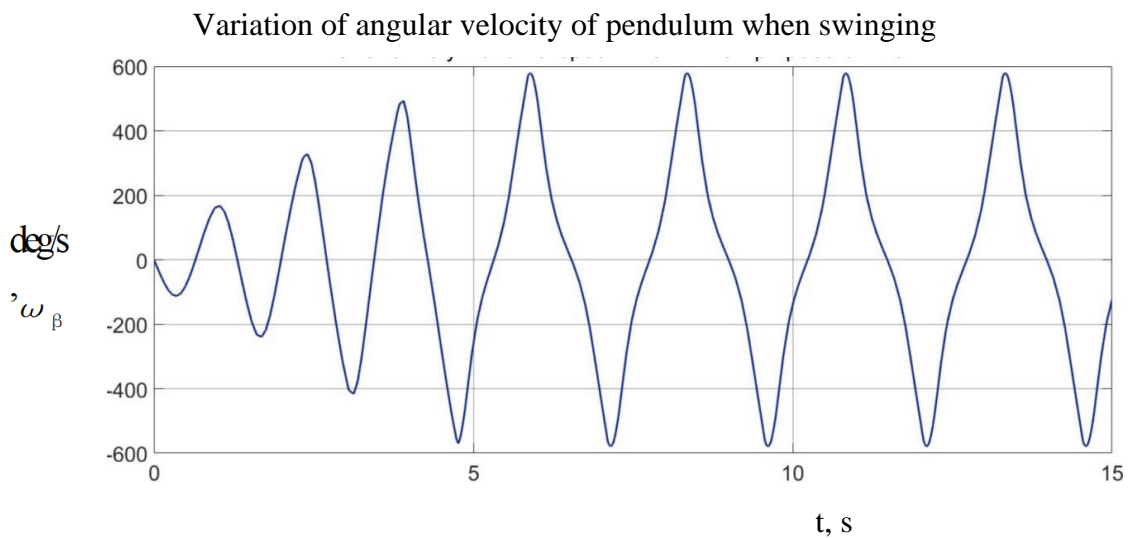
**Figure 8.** Changing the angle of rotation of the arm with the actuator



**Figure 9.** Changing the pendulum swing angle



**Figure 10.** Variation of angular velocity of the powered arm



**Figure 11.** Variation of the angular velocity of the pendulum

Now let's add a switch to the second regulator with stabilisation, when the swing regulator swings the pendulum to 135. The simulation results with the addition of a regulator to stabilise the pendulum are shown according to Figures 12-16.

Changing the angle of arm swing and stabilisation

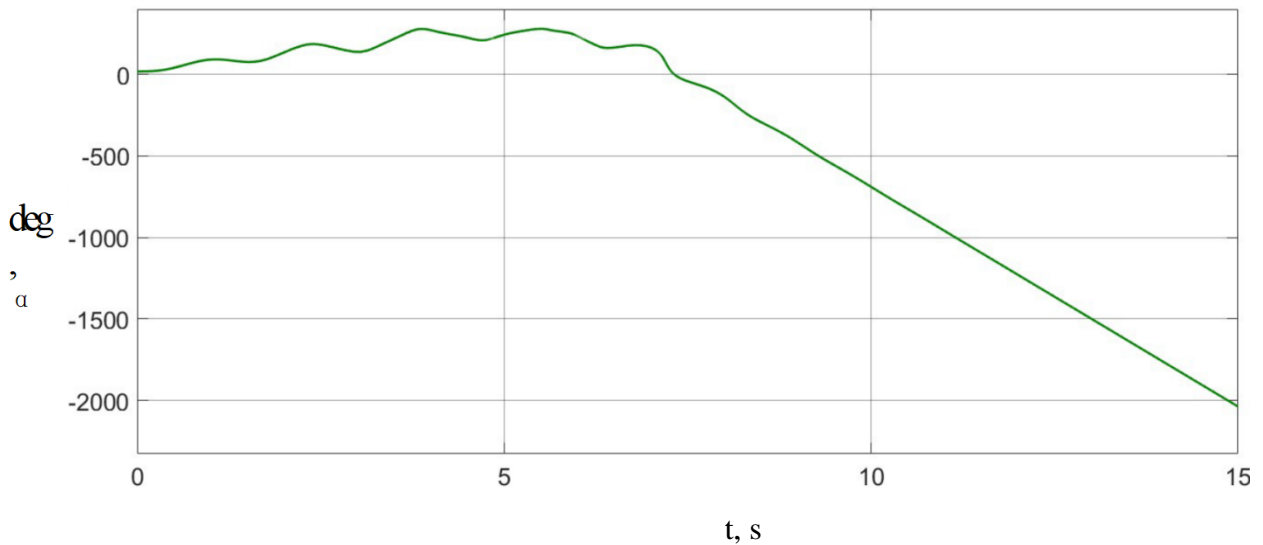


Figure 12. Variation of arm swing angle with actuator

Variation of the pendulum swing angle during swinging and stabilisation

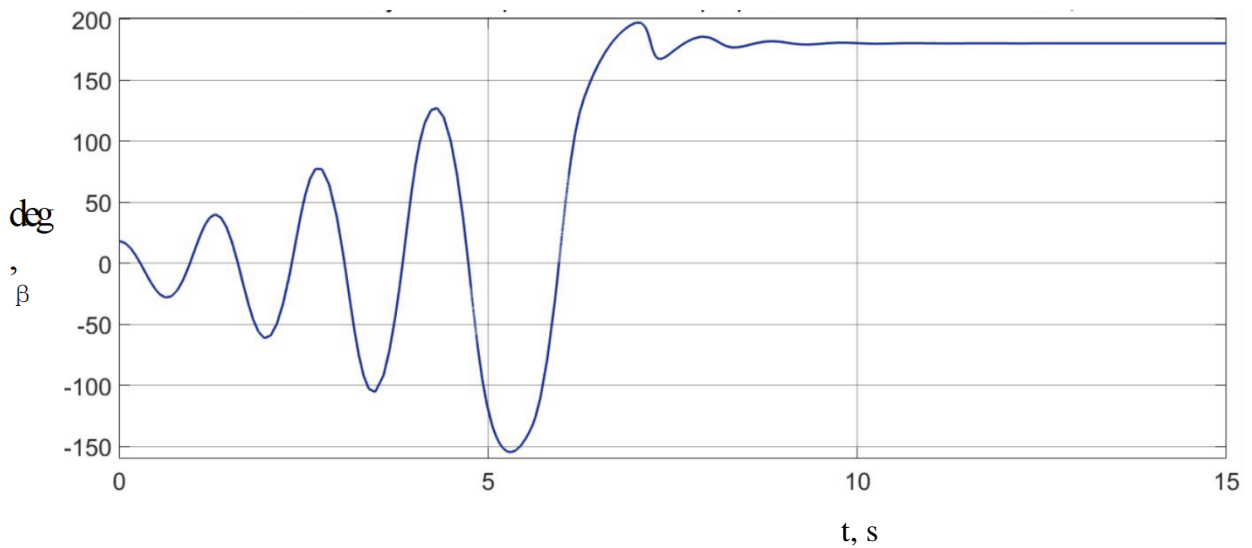
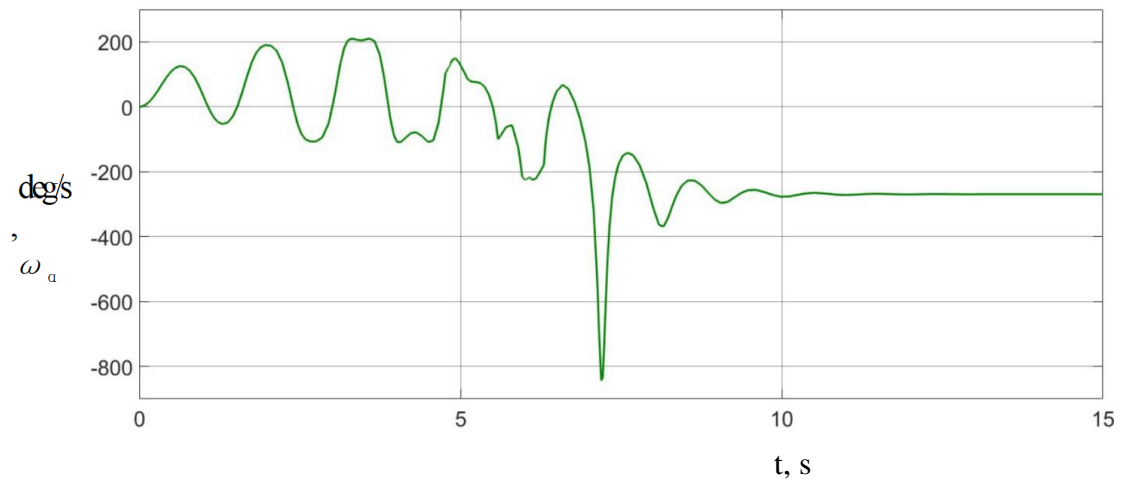


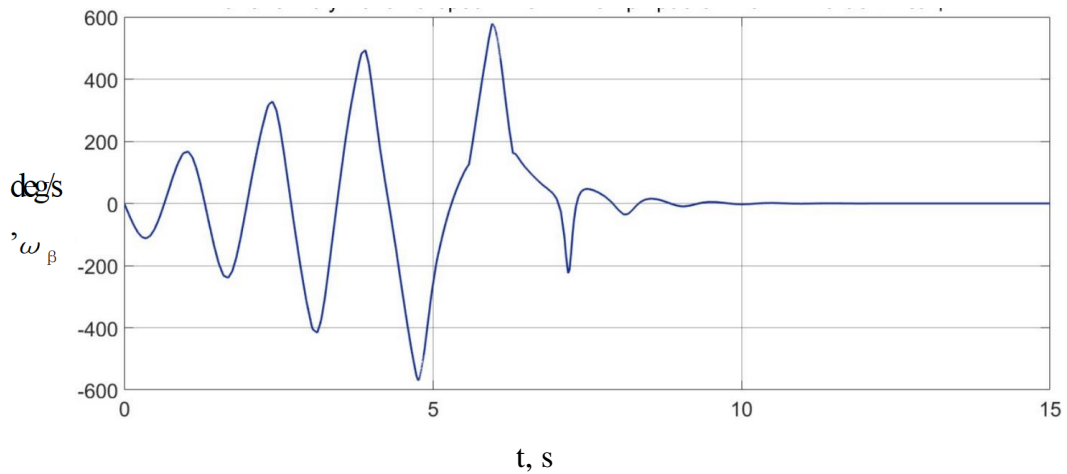
Figure 13. Изменение угла поворота маятника

Increasing the conditional speed of the arm during swinging and stavilivanni



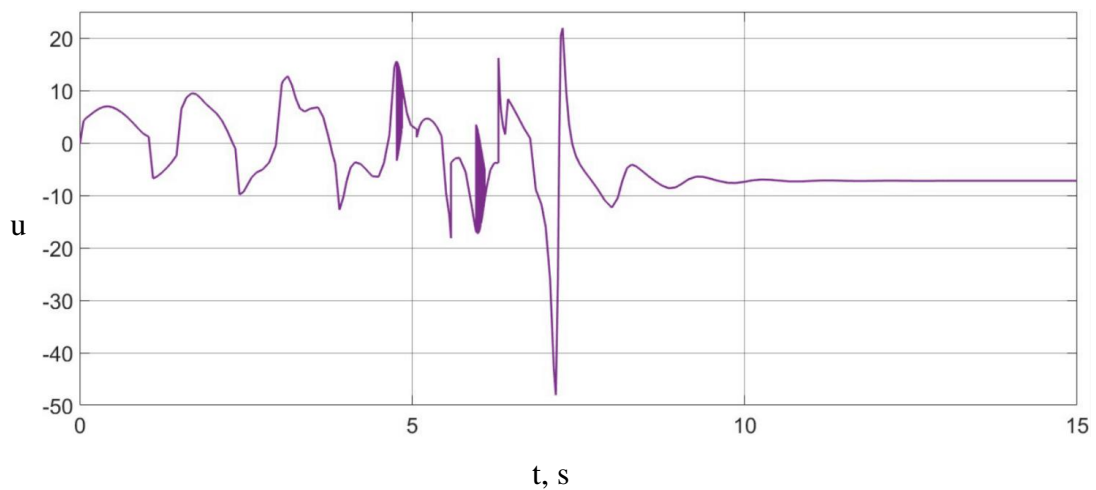
**Figure 14.** Variation of the angular velocity of the arm with the actuator

Variation of angular velocity of the pendulum during swinging and stabilisation



**Figure 15.** Variation of angular velocity of the pendulum

Variation of the control signal during swing and stabilisation



**Figure 16.** Variation of the system control signal

The simulation results prove the correctness of the synthesised algorithm for controlling the Furuta pendulum. As expected, the pendulum is balanced in the vertical position and the arm with the actuator rotates with a constant angular velocity, which indicates the stability of the system.

## 6. COUCLUTION

In this paper, the study of partially driven systems is carried out, analysing their properties, mathematical models, and control methods. The Furuta pendulum, also known as a pivoting inverted pendulum, was studied as an example of an underdriven system. In this paper, a model of the system is compiled, a control algorithm is synthesised, and mathematical modelling is carried out in Simulink MATLAB as a result of work on the control algorithm, two regulators were synthesised One controller, built using the energy approach, swings the pendulum to the stabilisation region. The second controller, built using feedback linearisation and PD controller methods, takes control of the system when the pendulum swings to  $135^\circ$  The synthesised controller for stabilising the Furuta pendulum provides stability of the angular velocity  $\alpha$  of the actuated arm, but not asymptotic stability. In the development of this work, it is planned to refine the regulator to provide asymptotic stability of the system. It is also planned to apply other approaches to the control of underdriven systems to improve dynamic performance.

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